TURING

WILEY

谨以此书纪念图灵诞辰百年

图灵的秘密

他的生平、思想及论文解读

The Annotated Turing

[美] Charles Petzold 著

杨卫东 朱皓 等 译



人民邮电出版社
POSTS & TELECOM PRESS

[illegible]•[illegible]……[illegible]

[illegible]Windows[illegible]Charles Petzold[illegible] [illegible]•[illegible] “On Computable Numbers, with an Application to the Entscheidungs problem”[illegible] [illegible] [illegible]41[illegible]•[illegible]

[illegible]•[illegible]1912-1954)

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]1966[illegible]

Charles Petzold

Windows[illegible]MVP[illegible]25[illegible]Windows[illegible]1994[illegible]5[illegible]Petzold[illegible]*Window Magazine*[illegible]Windows[illegible]7[illegible]Windows GDI[illegible]Win32 API[illegible]Windows[illegible]

[illegible]

◆ 1966 A. J. Perlis
[illegible]
◆ 1967 Maurice V. Wilkes
[illegible]EDSAC
◆ 1968 Richard W. Hamming
[illegible]
◆ 1969 Marvin Minsky
[illegible]
◆ 1970 J. H. Wilkinson
[illegible]
◆ 1971 John McCarthy
[illegible]
◆ 1972 Edsger W. Dijkstra
[illegible]
◆ 1973 Charles W. Bachman
[illegible]
◆ 1974 Donald E. Knuth
[illegible]“[illegible]”[illegible]
◆ 1975 Allen Newell[illegible]Herbert A. Simon
[illegible]
◆ 1976 Michael O. Rabin[illegible]Dana S. Scott
[illegible]
◆ 1977 John Backus
[illegible]
◆ 1978 Robert W. Floyd
[illegible]
◆ 1979 Kenneth E. Iverson
[illegible]APL
◆ 1980 C. Antony R. Hoare
[illegible]

◆ 1981 Edgar F. Codd

◆ 1982 Stephen A. Cook

NP

◆ 1983 Dennis M. Ritchie Kenneth L. Thompson

UNIX

◆ 1984 Niklaus E. Wirth

EULER ALGOL-W MODULA PASCAL

◆ 1985 Richard M. Karp

◆ 1986 John E. Hopcroft Robert E. Tarjan

◆ 1987 John Cocke

◆ 1988 Ivan E. Sutherland

◆ 1989 William V. Kahan

◆ 1990 Fernando J. Corbato

Multics

◆ 1991 Robin Milner

LCF) ML CCS)

◆ 1992 Butler W. Lampson

◆ 1993 Jurlis Hartmanis Richard E. Stearns

◆ 1994 Raj Reddy Edward Feigenbaum

◆ 1995 Manuel Blum

◆ 1996 Amir Pnueli

◆ 1997 Douglas Engelbart

◆; 1998 James Gray

◆ 1999 Frederick P. Brooks, Jr.

[illegible]

◆ 2000 [illegible]Andrew Chi-Chih Yao)

[illegible]

◆ 2001 Ole-Johan Dahl[illegible]Kristen Nygaard

[illegible]

◆ 2002 Ronald L. Rivest[illegible]Adi Shamir[illegible]Leonard M. Adelman

[illegible]RSA)

◆ 2003 Alan Kay

[illegible]Smalltalk

◆ 2004 Vinton G. Cerf[illegible]Robert E. Kahn

[illegible]

◆ 2005 Peter Naur

Algol 60[illegible]

◆ 2006 Frances E. Allen

[illegible])

◆ 2007 Edmund M. Clarke[illegible]Allen Emerson[illegible]Joseph Sifakis

[illegible]

◆ 2008 Barbara Liskov

[illegible]

◆ 2009 Charles P. Thacker

[illegible]Alto[illegible]

◆ 2010 Leslie L. Valiant

[illegible]

◆ 2011 Judea Pearl

[illegible]

□□□□□□□CIP□□□

□□□□□□□□□□□□□□□□□□ / □□□□□□□□Petzold, C.□ □□□□□□□. — □□ □□□□□□□□□ 2012.
11

□□□□□The Annotated Turing
ISBN 978-7-115-28214-9

I. ①□…   II. ①□… ②□…   III. ①□□□1912~1954□ —□□□□   IV. ①K835.616.16

□□□□□□□CIP□□□□□2012□□119110□

[illegible]

[illegible]·[illegible], [illegible] [illegible]

[illegible] [illegible]

[illegible]

◆ [illegible] [[illegible]] Charles Petzold

[illegible] [illegible] [illegible] [illegible] [illegible]

[illegible] [illegible]

◆ [illegible]   [illegible]14[illegible]

[illegible] 100061   [illegible] 315@ptpress.com.cn

[illegible] http://www.ptpress.com.cn

[illegible]   [illegible]

◆ [illegible]: 700 × 1000   1/16

[illegible]: 22.5

[illegible]: 428[illegible] 2012[illegible]11[illegible]1[illegible]

[illegible]: 1 -4 000[illegible] 2012[illegible]11[illegible]1[illegible]

[illegible] [illegible]01-2009-5735[illegible]

ISBN 978-7-115-28214-9

[illegible]69.00[illegible]

[illegible]010[illegible]51095186[illegible]604   **[illegible]010[illegible]67129223**

[illegible]010[illegible]67171154

[illegible]

[illegible]Türing Machine[illegible]1936[illegible]·[illegible]ü[illegible]

——[illegible] *Faster Than Thought: A Symposium on Digital Computing Machines*[illegible]1953[illegible]

[illegible]

[illegible]“[illegible]”[illegible]1936[illegible]·[illegible]1912—1954[illegible]——[illegible]

[illegible]

[illegible]

[illegible]“[illegible]”[illegible]·[illegible]“On Computable Numbers, with an Application to the Entscheidungsproblem”[illegible]“[illegible]”[illegible]“shy”[illegible]70[illegible]

[illegible]36[illegible][1] “On Computable Numbers, with an Application to the Entscheidungsproblem”[illegible]3[illegible][2][illegible]“[illegible]”[illegible]“[illegible]”[illegible]1937[illegible][3][illegible]“[illegible]”[illegible]

[illegible]

[illegible]

We shall avoid confusion by speaking more often of computable sequences than of computable numbers.

[illegible]

[illegible][4][illegible]

When the letters are replaced by figures, as in § 5, we shall have a numerical

[243]

description of the complete configuration, which may be called its description number.

[illegible]§5[illegible]

[illegible]

[illegible]

[illegible]4~15[illegible]16[illegible]11[illegible]

[illegible]·[illegible]1862—1943[illegible]“[illegible]”[illegible]60%[illegible]

[illegible]“[illegible]”[illegible]

[illegible]“20[illegible]”[5][illegible]

[illegible]

[illegible]“[illegible]”[illegible]

[illegible]“[illegible]”[illegible]

[illegible]“[illegible]”[illegible]

[illegible]“[illegible]”[illegible]

[illegible]

[illegible]

[illegible]·[illegible]30[illegible][6][illegible]“On Computable Numbers”[illegible]“[illegible]”[illegible]“Computing Machinery and Intelligence”[illegible]“[illegible]”[illegible]1950[illegible]“[illegible]”[illegible]

[illegible]·[illegible]

[illegible]

[illegible]“[illegible]”[illegible]41[illegible]

[illegible]·[illegible]1949—[illegible] *Alan Turing: The Enigma*[illegible]·[illegible]Simon & Schuster[illegible]1983[illegible]20[illegible]70[illegible]·[illegible]*Breaking the Code*[illegible]1986[illegible]1996[illegible]·[illegible]·[illegible]

[illegible]·[illegible]1791—1871[illegible]·[illegible]1815—1852[illegible]10[illegible]“[illegible]”[illegible]Pascal[illegible]“[illegible]”[illegible]

[illegible]“[illegible]”[illegible]*The Turing Omnibus*[illegible]1989[illegible]·[illegible]“[illegible]”[illegible]*Turing’s Man*[illegible]1984[illegible]·[illegible]*Ad Infinitum*[illegible]1993[illegible]*The Ghost in Turing’s Machine*[illegible]

[illegible]·[illegible]*Novel Gazing: Queer Readings in Fiction*[illegible]·[illegible] *The “Sinister Fruitiness” of Machines: Neuromancer, Internet Sexuality, and the Turing Test*[illegible]“[illegible]”[illegible]*Neuromancer*[illegible]·[illegible]“[illegible]”[illegible]*Neuromancer*[illegible]

[illegible]·[illegible]·[illegible]*The Turing Option*[illegible]1992[illegible]·[illegible]*Turing*[illegible]2003[illegible]

[illegible]·[illegible]*Turing’s Delirium*[illegible]·[illegible]·[illegible]2006[illegible]·[illegible]*A Madman Dreams of Turing Machines*[illegible]Knopf[illegible]2006[illegible]·[illegible]·[illegible]

[illegible]·[illegible]·[illegible]*Cryptonomicon*[illegible]Avon[illegible]1999[illegible]·[illegible]*Enigma*[illegible]Hutchinson[illegible]

1995[illegible]·[illegible]*The Cambridge Quintet: A Work of Scientific Speculation*[illegible]Perseus[illegible]1998[illegible]·[illegible]*Gödel, Escher, Bach*[7][illegible]Basic[illegible]1979[illegible]·[illegible]*The Turing Test*[illegible]BBC[illegible]2000[illegible]·[illegible]Doctor Who[illegible]

[illegible]·[illegible]

[illegible]

* * *

[illegible]1999[illegible]2004[illegible]2005[illegible]11[illegible]7[illegible]2007[illegible]2008[illegible]

[illegible]·[illegible]

[illegible]·[illegible]“On Computable Numbers, with an Application to the Entscheidungsproblem”[illegible]

[illegible]·[illegible]·[illegible]

[illegible]Wiley[illegible]· [illegible]· [illegible]·[illegible]·[illegible]Wiley[illegible]

[illegible]

[illegible]JSTOR[illegible]Wolfram MathWorld[illegible]

* * *

[illegible]www.TheAnnotatedTuring.com[illegible]

[illegible]·[illegible]

[illegible]

2008[illegible]5[illegible]

[1] □□·□□□“On Computable Numbers, with an Application to the Entscheidungsproblem”□*Proceedings of the London Mathematical Society*□2nd series□Vol.42□1936□□pp.230-265□

[2] □□·□□□“On Computable Numbers, with an Application to the Entscheidungsproblem A Correction”□*Proceedings of the London Mathematical Society*□2nd series□Vol.43□(1937)□pp.544-546□

[3] □□□·□□□ “On Computable Numbers, with an Application to the Entscheidungsproblem”□□□□□□□*The Journal of Symbolic Logic*□Vol. 2□No. 1□1937□3□□42-43□

[4] □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□——□□□

[5] □□□□□John P. Burgess□George S. Boolos□John P. Burgess□Richard C. Jeffrey□□□ *Computability and Logic*, fourth edition□Cambridge University Press□2002□□□□□□□□□

[6] □□□□□□□□□□□*The Collected Works of A.M. Turing*□Amsterdam: Elsevier□1992□2001□□4□□□□□□□□□□□□□□□B. Jack Copeland□□□ *The Essential Turing*□Oxford University Press□2004□□ *Alan Turing’s Automatic Computing Engine*□Oxford University Press□2005□□□□□□□□□□□□□□□□□□□□□□□□□20□□40□□□□ACE□□□□□□□

[7] □□□□□□□□□□□□□□——□□□□□□□□□□□□□1996□8□□□□

——□□□

□ □

[illegible]1[illegible]
[illegible]

[illegible]*Arithmetica*[illegible][1]

> [illegible][2]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]x[illegible]x[illegible]6[illegible]5[illegible]x[illegible]

$$\frac{x}{6}+\frac{x}{12}+\frac{x}{7}+5+\frac{x}{2}+4=x$$

[illegible]84[illegible]84[illegible]

$$14x+7x+12x+420+42x+336=84x$$

[illegible]x[illegible]

$$84x-14x-7x-12x-42x=420+336$$

[illegible]

$$9x=756$$

[illegible]

$$x=84$$

[illegible]14[illegible]7[illegible]12[illegible]33[illegible]5[illegible]42[illegible]80[illegible]4[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]7[illegible]12[illegible]

[illegible]2[illegible]6[illegible]12[illegible]7[illegible]84[illegible]

[illegible]84[illegible]150[illegible]280[illegible][3][illegible]295[illegible][4][illegible]276—[illegible]195[illegible]62[illegible]100—170[illegible]

[illegible]250[illegible]1893~1895[illegible]·[illegible]“[illegible]”[illegible]232~247[illegible]248~265[illegible][5][illegible]370—415[illegible]“[illegible]”[illegible]

[illegible]“[illegible]”[illegible]

[illegible]6[illegible]13[illegible]

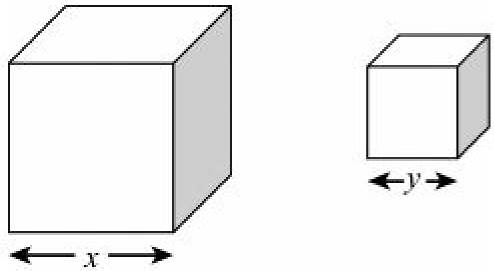
[illegible]4[illegible]“[illegible]-[illegible]”[illegible]5[illegible]“[illegible]-[illegible]”[illegible]6[illegible]“[illegible]-[illegible]”[illegible]

[illegible]4[illegible][6][illegible]

[illegible]

[illegible]

[illegible]370[illegible]10[illegible]



$$x+y=10$$

$$x^3+y^3=370$$

$$x^3+(10-x\}^3=370$$

$$x^3+(1000+30x^2-300x-x)^3=370$$

$$30x^2-300x+630=0$$

$$x^2-10x+21=0$$

$$(x-7)(x-3)=0$$

$$(5+x)^3+(5-x)^3=370$$

$$30x^2+250=370$$

$$x^2=4$$

[illegible]“[illegible]100[illegible]101[illegible]”[8]

[illegible]370[illegible]10[illegible]“[illegible]”[9][illegible]0[illegible]0[illegible]

[illegible]

$$\frac{3}{5}$$

[illegible]

[illegible]

> [illegible]8[illegible]5[illegible]60[illegible][10]

[illegible]“[illegible]”[illegible]25[illegible]5[illegible]5[illegible][11][illegible]5[illegible]

$$\frac{79}{12}$$

[illegible]8[illegible]

$$\frac{59}{12}$$

[illegible]5[illegible]79/12[illegible]8[illegible]59/12[illegible]$72\frac{1}{4}$[illegible]“[illegible]”[illegible]$72\frac{1}{4}$[illegible]

$$\frac{289}{4}$$

□□□□□□□□□□□□□□17□2□□□□□□□$72\frac{1}{4}$□$\frac{17}{2}$□□$8\frac{1}{2}$□□□□□□□□□□□□□□“□□□□□□□□60□□□□□□□□□□□□□□□□□□□□□□□□□” □□□□“□□”□□□□□□□□□□□□□□□□□□□□□□□□□□□·□□□□□□□□□□□□□□□□□60□$72\frac{1}{4}$□$132\frac{1}{4}$□□□□□□□□

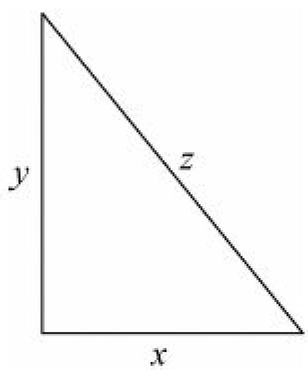
$$\frac{529}{4}$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□23□2□□□□□□□□□□□□□23/2□□$11\frac{1}{2}$□□□□□□□□□□79/12□59/12□□□□□

□□□□□□□□□□□□□□□□2□□□□8□□□□□□□□□□□□□□□□□□□□□□□□□□□x□y□z□□□□□□□

$$x^2+y^2=z^2$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□



□□□□□□□□□□□□□x□y□z□□□□3□4□5□□□□□□9□16□□□□25□□□□□□□□□□□□□□□□□□□□□□□□□□□“□□□□□□”□□□□z^2□□□16□□□□□□□□□□□144/25□256/25□□□□□□□□□□□□□□□□□□□□□□□□□□□12/5□□□□□□□□□16/5□□□□□□□□□□□4□□□□

$$\left(\frac{12}{5}\right)^2+\left(\frac{16}{5}\right)^2=4^2$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□5^2□□25□□□□□□□

$$12^2+16^2=20^2$$

□144□256□□400□□□4□□□□□4□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□20□□□□□□□12□16□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□3□

□□□□□□□□□□□□□4□□□□□□□□□□□□□□□□□□□$\frac{3}{4}$□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□1575□□□□□□□□□□□1621□□□□□□□□□1601—1665□□
□□□1621□□□□□□□□□□□□□□□□□□1670□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□

> □□□□□□□□□□□2□□□□□□□□4□□□□□□4□□□□□
> □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
> □□□□□□□□□□□□□□□□□□[12]

□□□□□□

$$x^3+y^3=z^3$$

□□□□□□□□□□4□5□6□□□□□□□□□□□□□□□□□□

$$x^3+y^3+1=z^3$$

□□□□

$$x^3+y^3=z^3$$

□□□□□□□□□□x□y□z□□□□6□8□9□□□

$$x^3+y^3-1=z^3$$

□□□□□□□□□□□□9□10□12□□□□□□□□□□□□□□

$$x^3+y^3=z^3$$

□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□·□□□1953—□□10□□□□
□□□□□□□□□□□□□1995□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□3□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□1□□□2□3□□
□□

[illegible]
[illegible]

[illegible]

□1□ □□□·□□□*Diophantus of Alexandria: A Study in the History of Greek Algebra*□second edition□Cambridge University Press□1910□Dover Publications□1964□□3□

□2□ *Greek Mathematical Works II: Aristarchus to Pappus of Alexandria*□Loeb Classical Library No. 362□□□Ivor Thomas□□□Harvard University Press□1941□□512–3□

□3□ □□□□□□Simon Hornblower and Antony Sprawforth□eds.□*Oxford Classical Dictionary*□revised third edition□Oxford University Press□2003□□483□

□4□ □□□□□□□□□□□□□□□□□□Charles Coulston Gillispie□ed.□*Dictionary of Scientific Biography* □Scribners□1970□□

□5□ □□□*Diophantus of Alexandria*□2□note 2□□□□□□□□□□□□□□□□□

□6□ □□□*Diophantus of Alexandria*□168□

□7□ □□$ax^2 + bx + c = 0$□□□$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$□

□8□ Hermann Hankel (1874) □□□□□ *Diophantus of Alexandria*□54–55□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□Isabella Grigoryevna Bashmakova□*Diophantus and Diophantine Equations*□Mathematical Association of America□1997□□ch. 4□

□9□ □□□*Diophantus of Alexandria*□52–53□

□10□ □□□*Diophantus of Alexandria*□224□

□11□ □□□*Diophantus of Alexandria*□225□

□12□ □□□*Diophantus of Alexandria*□144□note 3□

□2□
□□□□□□□

□1□2□3…□□
□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□
□□□
□□□□

… -3 -2 -1 0 1 2 3 …

□□□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□□0□1□2□3□…□□
□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$\frac{3}{5}$$

□□□□□□□□□□□□□□□□□□

0.6

□□□□□□□□□□□□□□□□□□□47□□□□□□□□□1□□□□□□

$$\frac{47}{1}$$

□□□□□□□□□□□□□□□

-23.45678

□□□□□□□□□

$$\frac{-2345678}{100000}$$

□□□□□□□□□

$$\frac{1}{3}$$

□□□□□□□□□□□□□□□

0.3333333333…

□□□

$$0.234562345623456\ldots$$

[illegible]23456[illegible]x[illegible]

$$x=0.234562345623456\ldots$$

[illegible]100000[illegible]

$$100000x=23456.23456234562346\ldots$$

[illegible]10000x[illegible]23456.23456…[illegible]x[illegible]0.23456…[illegible]

$$99999x=23456$$

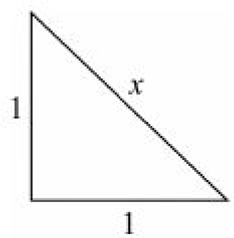
[illegible]

$$x=\frac{23456}{99999}$$

[illegible]

[illegible]

[illegible]



[illegible]

$$x^2=1^2+1^2$$

[illegible]

$$x^2=2$$

[illegible]

$$x=\sqrt{2}$$

[illegible]2[illegible]

$$\frac{53492}{37825}$$

[illegible]1.999 95[illegible]

[illegible]

[illegible]

[illegible]“[illegible]”[illegible]

[illegible]2[illegible]a[illegible]b[illegible]

$$\frac{a}{b}=\sqrt{2}$$

a[illegible]b[illegible]2[illegible]a[illegible]b[illegible]2[illegible]a[illegible]b[illegible]

[illegible]

$$\frac{a^2}{b^2}=2$$

[illegible]

$$a^2=2b^2$$

[illegible]a[illegible]b[illegible]2[illegible]a[illegible]a[illegible]a[illegible]a[illegible]b[illegible]b[illegible]

[illegible]a[illegible]2[illegible]c[illegible]

$$(2c)^2=2b^2$$

[illegible]

$$4c^2=2b^2$$

[illegible]

$$2c^2=b^2$$

[illegible]b[illegible]b[illegible]“a[illegible]b[illegible]”[illegible]

[illegible]“2[illegible]”[illegible]2[illegible]

1.4142135623730950488016887242097…

[illegible]6[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]surds[illegible]“[illegible]”[illegible]absurd[illegible]6[illegible]2[illegible]

[illegible]

[illegible]

.0010110111011110111110111111...

[illegible]0[illegible]1[illegible]0[illegible]1[illegible]0[illegible]1[illegible]

2[illegible]

$$x^2-2=0$$

[illegible]2[illegible]17[illegible]

$$x^3-17=0$$

[illegible]

$$-12x^5+27x^4-2x^2+8x-4=0$$

[illegible]x[illegible]1[illegible]-4[illegible]x[illegible]-12[illegible]27[illegible]-2[illegible]8[illegible]-4[illegible]“[illegible]”[illegible]x[illegible]

[illegible]

$$a_Nx^N+a_{N-1}x^{N-1}+\dots+a_2x^2+a_1x+a_0=0$$

[illegible]N[illegible]a_i[illegible]

$$\sum_{i=0}^{N} a_i x^i = 0$$

[illegible]

$$-12x^5 + 27x^4 - 2x^2 + 8x - 4 = 0$$

[illegible]N[illegible]5[illegible]a_5[illegible]-12[illegible]a_4[illegible]27[illegible]a_3[illegible]0[illegible]

[illegible]N[illegible]N[illegible]1[illegible]

$$x^2 - 10x + 21 = 0$$

[illegible]3[illegible]7[illegible]

2[illegible]

$$x^2 - 2 = 0$$

[illegible]2[illegible]

[illegible]5[illegible]

$$x - 5 = 0$$

[illegible]3/7[illegible]

$$7x - 3 = 0$$

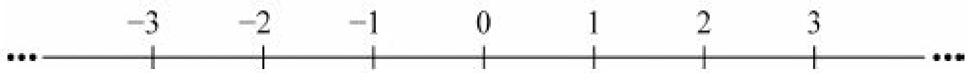
[illegible]

$$x^2 + 5 = 0$$

[illegible]5[illegible]0[illegible]-1[illegible] i[illegible]

[illegible]18[illegible]

[illegible]



[illegible]

[illegible]*a*[illegible]*b*[illegible]*a*[illegible]*b*[illegible]

$$\frac{a+b}{2}$$

[illegible]2[illegible]

[illegible]

[illegible]

1740[illegible]·[illegible]1707—1783[illegible]“[illegible]”[1][illegible]

[illegible]1844[illegible]·[illegible]1809—1882[illegible]30[illegible]

.110001000000000000000001000000…

[illegible]

1!=1

2!=1x2=2

3!=1x2x3=6

4!=1x2x3x4=24

5!=1x2x3x4x5=120

[illegible]1[illegible]2[illegible]6[illegible]24[illegible]120[illegible]…[illegible]1[illegible]0[illegible][2][illegible]

1882[illegible]·[illegible]1852—1939[illegible]π[illegible]

π=3.14159265358979323846264338327 95…

[illegible]π[illegible]“[illegible]”[illegible]“[illegible]”[illegible]τετραγωνίζειν[illegible]“[illegible]”[illegible][3]

[illegible]π[illegible]

[illegible]e[illegible]

$$\left(1+\frac{1}{N}\right)^{N}$$

[illegible]*N*[illegible]e[illegible]

e=2.71828182845904523536028747135 27…

[illegible]e[illegible]

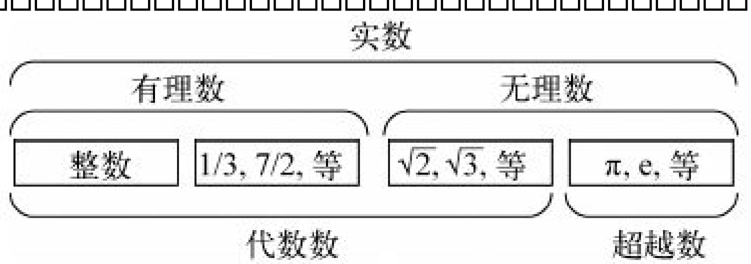
$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\cdots$$

[illegible]

[illegible]

π=π

[illegible]



[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]——[illegible]——[illegible]

[illegible]

{1,2,3,4}

[illegible]4[illegible]

{4,1,3,2}

[illegible]4[illegible]

[illegible]

{1,2,3,…}

[illegible]

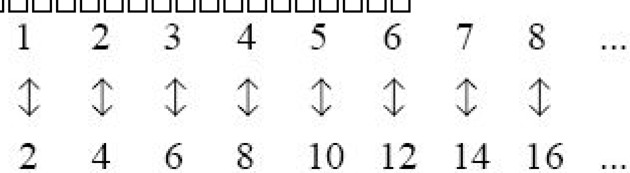
{2,4,6,…}

[illegible]

[illegible]

[illegible]“[illegible]”[illegible]“1[illegible]2[illegible]3[illegible]…”[illegible]

[illegible]



[illegible]1683[illegible][4][illegible]

[illegible]· [illegible]1845—1918[illegible]

[illegible]17[illegible]“[illegible]”[illegible][5][illegible]
[illegible]1869[illegible]

[illegible]1873[illegible]·[illegible]1831—1916[illegible]
[illegible]
[illegible]“[illegible]”[illegible][6][illegible]

[illegible]
[illegible]1[illegible]
2[illegible]3[illegible]…[illegible]

[illegible]
[illegible]

0

1

-1

2

-2

3

-3

…

[illegible]

[illegible]

1/1

1/2

2/1

1/3

2/2

3/1

1/4

2/3

3/2

4/1

…

[illegible]2[illegible]3[illegible]
[illegible]4[illegible]
[illegible]

[illegible]1874[illegible]“[illegible]”[7][illegible]
[illegible]

$$a_N x^N + a_{N-1} x^{N-1} + \ldots + a_2 x^2 + a_1 x + a_0 = 0$$

[illegible] N [illegible] a_i [illegible] a_i [illegible] N [illegible] 5 [illegible] N [illegible]

[illegible]

[illegible]

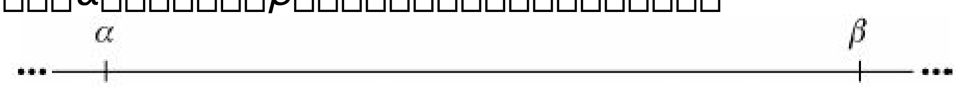
[illegible] 1874 [illegible]

[illegible] ω [illegible]

$$\omega_1\ \omega_2\ \omega_3\ \omega_4\ \omega_5\ \omega_6\ \ldots$$

[illegible] —— [illegible]

[illegible] α [illegible] β [illegible]



[illegible] α [illegible] β [illegible] α [illegible] β [illegible] α' [illegible] β'



[illegible] α' [illegible] β' [illegible] α'' [illegible] β''

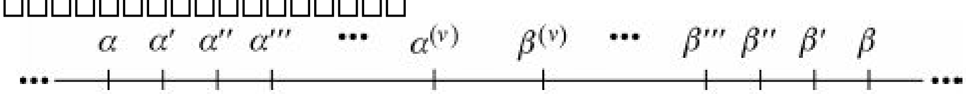


[illegible]



[illegible]

[illegible]



[illegible] (v) [illegible] v [illegible] $\alpha^{(v)}$ [illegible] $\beta^{(v)}$ [illegible] $\alpha^{(v)}$ [illegible] $\beta^{(v)}$ [illegible] $\alpha^{(v)}$ [illegible] $\beta^{(v)}$ [illegible]

$$\frac{\alpha^{(v)} + \beta^{(v)}}{2}$$

[illegible]

[illegible] α [illegible] β [illegible] α [illegible] β [illegible] α [illegible] β [illegible] α [illegible] β [illegible] α [illegible] β [illegible] α^{∞} [illegible] β^{∞} [illegible]

α^{∞} [illegible] β^{∞} [illegible]



[illegible] α [illegible] α^{∞} [illegible] β [illegible] β^{∞} [illegible] α^{∞} [illegible] β^{∞} [illegible]

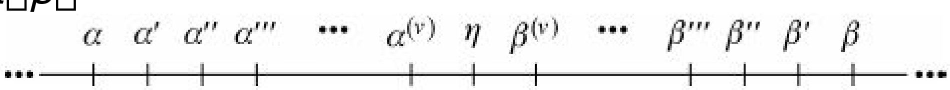
$$\frac{\alpha^{\infty} + \beta^{\infty}}{2}$$

α^{∞} [illegible] β^{∞} [illegible] η [illegible] beta [illegible]



[illegible] α [illegible] η [illegible] β [illegible] η [illegible]

[illegible] η [illegible] α [illegible] β [illegible] η [illegible] α [illegible] β [illegible]



[illegible] $\alpha^{(v)}$ [illegible] $\beta^{(v)}$ [illegible] η [illegible]

[illegible]——[illegible]

[illegible]

[illegible]

[illegible]

[illegible] 20 [illegible][8] [illegible]“[illegible]”[illegible]

1 0 1

1 0.5 2 0.25 4 0.1 10 0.0001

10000 0 1

M

x y 0 1 (x

y) x a

$$x=.a_1a_2a_3a_4\ldots$$

y

$$y=.b_1b_2b_3b_4\ldots$$

$$.a_1b_1a_2ba_2a_3b_3a_4b_4\ldots$$

[9] “Je le vois, mais je ne le crois pas.”

1891 [10]

diagonal proof diagonal process

diagonal argument diagonalization

0 1

.1234567890…

.2500000000…

.3333333333…

.3141592653…

.0101101110…

.4857290283…

.0000000000…
.9999999999…
.7788778812…
.2718281828…
…

[illegible]0[illegible]1/4[illegible]1/3[illegible]π/10[illegible]e/10[illegible]“1”[illegible]0[illegible]

[illegible]

.1234567890…
.2500000000…
.3333333333…
.3141592653…
.0101101110…
.4857290283…
.0000000000…
.9999999999…
.7788778812…
.2718281828…
…

[illegible]

.1531190918…

[illegible]1[illegible]9[illegible]0[illegible]

.2642201029…

[illegible]1[illegible]2[illegible]

[illegible]5[illegible]6[illegible]

[illegible]3[illegible]4[illegible]

[illegible]*N*[illegible]*N*[illegible]*N*[illegible]*N*[illegible]

[illegible]0[illegible]1[illegible]

[illegible]

[illegible]

1895[illegible]0[illegible]$\aleph_0$[illegible]“[illegible]”[illegible]$\aleph_1$[illegible]$\aleph_2$[illegible]$\aleph_3$[illegible]

[illegible]$\aleph_0$[illegible]

[illegible]3[illegible]

$$\{a, b, c\}$$

[illegible]3[illegible]

$$\{\ \} \qquad \{a, b\}$$
$$\{a\} \qquad \{a, c\}$$
$$\{b\} \qquad \{b, c\}$$
$$\{c\} \qquad \{a, b, c\}$$

[illegible]3[illegible]8[illegible]

$$2^3=8$$

[illegible]4[illegible]16[illegible]2[illegible]4[illegible]5[illegible]32[illegible]

[illegible]0[illegible]1[illegible]

a	b	c	子集
0	0	0	$\{\ \}$
0	0	1	$\{c\}$
0	1	0	$\{b\}$
0	1	1	$\{b, c\}$
1	0	0	$\{a\}$
1	0	1	$\{a, c\}$
1	1	0	$\{a, b\}$
1	1	1	$\{a, b, c\}$

□□□□□0□1□□□□□□□□□□0□7□□□□3□□□8□□□□□□□□□□□□□

□□□□ = 2□□□□□

□□□10□□□□□□□□□□□□1024□□□□□□□100□□□□□□□□□□□□1267650600228229401496703205376□□□□

□□□□□□□□□□□□□□0□□□□□□□

{0, 1, 2, 3, 4, 5, …}

□□□□□□□$\aleph_0$□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$2^{\aleph_0}$$

□□□0□1□□□□□□□□□□□□□□□□□□□□□□

0	**1**	**2**	**3**	**4**	**5**	…	□□
0	0	0	0	0	0	…	{ }
1	0	0	0	0	0	…	{0}
0	1	0	0	0	0	…	{1}
1	1	0	0	0	0	…	{0□1}
0	0	1	0	0	0	…	{2}
1	0	1	0	0	0	…	{0, 2}
0	1	1	0	0	0	…	{1, 2}

1 1 1 0 0 0 … {0, 1, 2}

… … … … … … … …

□□□□□□□□□□□□□□□□□0□1□□□□□□□□□□□□□□□
□□

.000000…
.100000…
.010000…
.110000…
.001000…
.101000…
.011000…
.111000…
…

□□□0□1□□□□□□□□□0□1□□□□□□□□□□□0□1□□□□□□ [11] □□□□□□□0□1□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□

□□□□□□□□

$$2^{\aleph_0}$$

□□ $\aleph_0$ □□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□1891□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□ $\aleph_0$ □□□□□□□□□□□□ $\aleph_1$ □□□□□□□□□□
□□□□□□□□□□

$$\aleph_1 = 2^{\aleph_0}$$

□□□□□□□□□□□□□□□□□□□□□ $\aleph_0$ □□□□□□□□□□□□
□□

□□□□□□□□□□□□□□□□□□□□□□□□□□

$$\aleph_0 < 2^{\aleph_0}$$

□□□□□□□□□□□□□□

$$\aleph_0 \quad < < < < < < < < < < < < < < < < < < < < < < < < \; 2^{\aleph_0}$$


这是已知的。
$\aleph_0 < < < < < < < < < < < < < < < < < < < < < < < < 2^{\aleph_0}$
这是我们需要做的事。

[1] “□□” □□□□ “oiler”□□□□□□ “Euler” □□□□□——□□□

[2] □□□□□Edward B. Burger□Robert Tubbs□□□*Making Transcendence Transparent: An Intuitive Approach to Classical Transcendental Number Theory*□Springer□2004□□9-26□□□□

[3] □□E.W. Hobson□*Squaring the Circle: A History of the Problem*□Cambridge University Press□1913□□3□

[4] □□□·□□□□*Two New Sciences* □□□□□□□□□2E□Stillman Drake□□Wall & Emerson□2000□□40-41□□□□□ *Opere di Galileo Galilei*□Florence□1898□□VIII□78-79□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□·□□·□□□□□□□□□□□*Mathematical Logic*□Wiley□1967□Dover□2002□□pg.176□□□121□

[5] □□□·□□·□□□*Georg Cantor*□*His Mathematics and Philosophy of the Infinite*□Harvard University press□1979□Princeton University Press□1990□□277□

[6] □□□·□□□□□1873□11□29□□□□□□□ *From Kant to Hilbert:A Sonrce Book in the Foundations of Mathematics*□Oxford University Press□1996□□Vol. II□844□

[7] □□ *From Kant to Hilbert*□Vol. Ⅱ□839-843□□□

[8] □□□*Georg Cantor*□285□

[9] 1877□6□29□□□□*From Kant to Hilbert*□Vol. Ⅱ□860□

[10] □□□□·□□□□“□□□□□□□□□□□□□□”□*From Kant to Hilbert*□Vol. Ⅱ□920-922□

[11] □□□□□□□□□□□□□□□□□□□□□□□0□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□0□1□□□

[illegible]3[illegible]
[illegible]

[illegible]1999[illegible]12[illegible]31[illegible]75[illegible]1975[illegible]99[illegible]00[illegible]Y2K[illegible]

[illegible]1998[illegible]*Y2K: The Day the World Shut Down*[illegible]*Deadline Y2K*[illegible]*Y2K: It's Already Too Late*[illegible]*Y2K: An Action Plan to Protect Yourself, Your Family, Your Assets, and Your Community on January 1, 2000*[illegible]*101 Ways to Survive the Y2K Crisis*[illegible]*Y2K for Women: How to Protect Your Home and Family in the Coming Crisis*[illegible]*Crisis Investing for the Year 2000: How to Profit from the Coming Y2K computer Crash*[illegible]*Y2K: A Reasoned Response to Mass Hysteria*[illegible]*Spiritual Survival During the Y2K Crisis*[illegible]*Y2K: The Millennium Bug – A Balanced Christian Response*[illegible]*Awakening: The Upside of Y2K*[illegible]*Y2K-9: The Dog Who Saved the World*[illegible]1999[illegible]4[illegible]"Lights Out"[illegible]"Learning to Love Y2K"[illegible][1]

[illegible]

20[illegible]

[illegible]19[illegible](1824—1907[illegible]

[illegible]19[illegible]

19[illegible]300[illegible]oiler[illegible]yoo-clid[illegible]

[illegible]

[illegible]

1. [illegible]

2. [illegible]

3. [illegible]

4. [illegible]

[illegible]

5. [illegible][2]

[illegible]

[illegible]

[illegible]19[illegible]

[illegible]·[illegible]·[illegible]1777—1855[illegible]·[illegible]1802—1860[illegible]·[illegible]·[illegible]1792—1856[illegible]

[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

19[illegible][3][illegible]

[illegible]

[illegible]“[illegible]”[4]

[illegible]·[illegible]1862—1943[illegible]

[illegible]·[illegible]1724—1804[illegible]1895[illegible]9[illegible]“[illegible]”[5]

[illegible]1833[illegible]·[illegible]1804—1891[illegible]·[illegible]1849—1925[illegible]

[illegible]1898[illegible]1899[illegible]

[illegible]1899[illegible]*Grundlagen der Geometrie*[illegible]2[illegible]10[illegible]Open Court[illegible]1971[illegible]

[illegible]

[illegible]1889[illegible]·[illegible]1858—1932[illegible]——[illegible]· [illegible]1848—1925[illegible]1879[illegible]*Begriffsschrift*[illegible][6][illegible]*Grundlagen der Arithmetik*[illegible]1884[illegible]1893[illegible]*Grundgesetze der Arithmetik*[illegible]

[illegible]1900[illegible]8[illegible]

[illegible]·[illegible]1864—1909[illegible]

[illegible]

[7]

[illegible]1900[illegible]8[illegible]8[illegible]

[illegible][8]

[illegible]

[illegible]……[illegible]ignorabimus[illegible]

[9]

[illegible]ignorabimus[illegible]“[illegible]”[illegible]1876[illegible]·[illegible]—[illegible]1818—1896[illegible]—[illegible]“[illegible]……[illegible]”[10]

[illegible]—[illegible]“[illegible]”[illegible]

[illegible]23[illegible]10[illegible]23[illegible]

[illegible]“[illegible]”[illegible]·[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible][11]

[illegible]

Entscheidung der Lösbarkeit einer
diophantischen Gleichung.

[illegible]Entscheidung[illegible]

10. [illegible]

[illegible][12]

[illegible]1650[illegible]

[illegible]

Algorithm[illegible]Algorithmus[illegible]20[illegible]60[illegible][13]

[illegible]1900[illegible]“[illegible]”20[illegible]“[illegible]”[illegible]1905[illegible]·[illegible]1879—1955[illegible]“[illegible]”[illegible]annus mirabilis[illegible]

[illegible]1907[illegible]· [illegible]Zaumreit[illegible][14][illegible]1902[illegible]1909[illegible]1927[illegible]

[illegible]“[illegible]”[illegible]

[illegible]·[illegible]1913[illegible]20[illegible]20[illegible]·[illegible]

20[illegible]1902[illegible]

[illegible]·[illegible]1848[illegible]20[illegible]44[illegible]1893[illegible]1902[illegible]

[illegible]

[illegible]·[illegible]1872—1970[illegible]·[illegible]1792—1878[illegible][15][illegible]·[illegible]·[illegible]1806—1873[illegible]

> 11[illegible]
> [16]

1902[illegible]

[illegible]

[illegible]

[illegible][17][illegible]

[illegible]·[illegible]1908[illegible][18][illegible]1[illegible]1[illegible]2[illegible]2[illegible]3[illegible]

[illegible]1908[illegible]·[illegible]·[illegible]1861—1947[illegible]*A Treatise on Universal Algebra*[illegible]1898[illegible]1906[illegible]

[illegible]·[illegible]·[illegible]·[illegible]*Principia Mathematica*[illegible]1910[illegible]1912[illegible]1913[illegible]

[[illegible]·[illegible]1687[illegible]*Philosophiae Naturalis Principia Mathematica*[illegible]][illegible]·[illegible]·[illegible]1873—1958[illegible]*Principia Ethica*[illegible]1903[illegible][19][illegible]“[illegible]”[illegible]

[illegible]“[illegible]”[illegible][20]

[illegible]

1 + 1 = 2

[illegible][21]

[illegible]·[illegible]1904[illegible]“[illegible]”[illegible]

[illegible]1914[illegible]8[illegible]4[illegible]1918[illegible]

[illegible]1914[illegible]“[illegible]”[illegible][22][illegible]1916[illegible][23]

[illegible]1917[illegible][24]

1917[illegible]9[illegible]11[illegible]“[illegible]”[illegible]20[illegible]20[illegible]“[illegible]”[illegible] “[illegible]”[illegible]

[illegible]

〉[illegible]

〉[illegible]

〉[illegible]

〉[illegible]

[illegible]

[illegible]

[illegible]（well-formed formula[illegible]wff[illegible]woof[illegible]

[illegible]A[illegible]

$$gobbledygook = yadda–yadda–yadda$$

[illegible]B[illegible]

$$gobbledygook \neq yadda–yadda–yadda$$

[illegible]A[illegible]B[illegible]A[illegible]

[illegible]A[illegible]B[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]A[illegible]B[illegible]——[illegible]（ex falso quodlibet[illegible]

[illegible]

[illegible]A[illegible]B[illegible]A[illegible]B[illegible]

[illegible]2[illegible]“[illegible]”[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]——[illegible]

[illegible]A[illegible]B[illegible]A[illegible]B[illegible]——[illegible]

[illegible]A[illegible]B[illegible]

[illegible]1900[illegible]1917[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible][25][illegible]“[illegible]”[illegible]

[illegible]Entscheidung[illegible]Problem[illegible]·[illegible]1891—1970[illegible]1921[illegible]5[illegible]10[illegible]“Entscheidungsproblem und Algebra der Logik”[illegible]

> [illegible][26]

[illegible]

[illegible]1922[illegible]1923[illegible]“[illegible]”[illegible]Entscheidungsproblem[illegible][27][illegible]1928[illegible]·[illegible]1896—1962[illegible]1917[illegible]1918[illegible]*Grundzüge der theoretischen Logik*[28][illegible]

[illegible]120[illegible]engere Funktionenkalkül[illegible]“[illegible]”[illegible]

[illegible]·[illegible]1906—1978[illegible]

> [illegible][29]

[illegible]“[illegible]”[illegible]1929[illegible]

[illegible]

[illegible]

1930[illegible]·[illegible]68[illegible]“[illegible]”[illegible][30]30[illegible]“[illegible]”[illegible]“[illegible]ignorabimus[illegible]“[illegible]·[illegible]

> [illegible]
> ignorabimus[illegible]
> [illegible]
> [illegible]
> Wir müssen wissen. Wir werden wissen.

[illegible]1930[illegible]9[illegible]7[illegible]

[illegible]“[illegible]”[illegible]

[illegible]“[illegible]I”[illegible][31][illegible]I[illegible]

[illegible]

[illegible]·[illegible]“[illegible]”[illegible][32][illegible]

[illegible]·[illegible]

> [illegible][33]

[illegible]1950[illegible]“[illegible]”[34][illegible]

[illegible]20[illegible]20[illegible]·[illegible]·[illegible]1903—1957[illegible]

[illegible]1933[illegible]·[illegible]1888—1972[illegible]4[illegible]·[illegible]1885—1955[illegible]·[illegible]1888—1977[illegible]1934[illegible]1939[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[35]

[illegible]1928[illegible]“[illegible]”[illegible]*Entscheidungsproblem muß als das Hauptproblem der mathematischen Logik bezeichnet werden.*[illegible][36][illegible]

[illegible]9[illegible]“[illegible]”[illegible]

[illegible]1936[illegible]·[illegible]1903—1995[illegible]“[illegible]”[illegible]*The general case of the Entscheidungsproblem of the engere Funktionenkalkül* [first-order predicate logic] *is unsolvable.*[illegible][37]

[illegible]·[illegible]“[illegible]”[38][illegible]“[illegible]”[39]

[illegible]74[illegible][40][illegible]1943[illegible]

Wir müssen wissen.

Wir werden wissen.

[illegible]

[illegible]1945[illegible]

[illegible]——[illegible]20[illegible]2000[illegible]

1/1/00[illegible]

[illegible]2000

———————

[illegible]

[illegible]’00[illegible]

[illegible]

[illegible]“year”[illegible]“yr”[illegible]

[illegible]

[illegible]

[illegible]·[illegible]

---

[1]□□□□□7□□4□□1999□4□□□□□□www.wired.com/wired/archive/7.04□

[2] □□□·□□□□□□*The Thirteen Books of Euclid's Elements*□2E□Cambridge University Press□1926□Dover Publications□1956□□Vol. 1□154-155□

[3] □□□·□□□□□□*Mathematics in Aristotle*□Oxford University Press□1949□Thoemmes Press□1998□□50-57□□□□□Howard Eves□*Foundations and Fundamental Concepts of Mathematics*□3rd edition□PWS-Kent□1990□Dover□1997□□29-32□

[4] □□□□·□□□*Hilbert*□Springer-Verlag□1970□1996□□57□

[5] □□□*Hilbert*□25□

[6] □□□□Jean van Heijenoort□ed.□*From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*□Harvard University Press□1967□□1-82□

[7] □□□*Hilbert*□69□

[8] □□Ben H. Yandell□*The Honors Class: Hilbert's Problems and Their Solvers*□A. K. Peters□2002□□389□□□□□□□□□□□□ *Bulletin of the American Mathematical Society*□Vol. 8□July 1902□□□Mary Winston Newson□□□□□□□Jeremy J. Gray□*The Hilbert Challenge*□Oxford University Press□2000□□240□

[9] □□Yandell□*The Honors Class*□395□

[10] □□Gray□*The Hilbert Challenge*□58□

[11] □□Yandell□*The Honors Class*□397□

[12] □□Yandell□*The Honors Class*□406□□Ivor Grattan-Guinness□ "A Sideways Look at Hilbert's Twenty-three Problems of 1900"□*Notices of the American Mathematical Society*□Vol. 47□No. 7□August 2000□□752-757□□□□□ "□□□□□□□□□□□□□"□□□ "□□□□" □□□□□□□□□□□□□

[13] □*Oxford English Dictionary*□2nd edition□I□313□ □□Donald E. Knuth□□ "*The Art of Computer Programming*"□Vol. 1□Fundamental Algorithms□3rd edition□Addison-Wesley□1997□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□ "Euclid's Algorithm" □□20□□□□□□□□□

[14] □□□□□□□□□□□□*Raum*□*Zeit*□——□□□

[15] □□□□□□□□□□□□□□□□□——□□□

[16] □□□·□□□*The Autobiography of Bertrand Russell*□*1872-1914*□George Allen and Unwin Ltd□1967□□36□

[17] *From Frege to Gödel*□124-125□

[18] □□□·□□□“The Theory of Types” □ *From Frege to Gödel*□150-182□□

[19] I. Grattan-Guinness□*The Search for Mathematical Roots, 1870-1940: Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell to Gödel*□Princeton University Press□2000□□380□

[20] Grattan-Guinness□*The Search for Mathematical Roots*□454□

[21] □□□□□□□*54.43□□“From this proposition it will follow, when arithmetical addition has been defined, that 1 + 1 = 2”□□□□□□□□□□□□□□□□ *Principia Mathematica to*56*□Cambridge University Press□1997□□□□□□□□□□□□□□□□□□□□□□□□□□Vol. II *110.643□□□□□□“□□□□□□□□□□□”□

[22] □□□*Hilbert*□137□

[23] □□□□□G. H. □□□□□□□□□□□□□□□□□□□1942□□Cambridge University Press□□□□□□□□□□□□□□□□□□□□□□□□□□□□ *Bertrand Russell and Trinity: A College Controversy of the Last War*□Cambridge University Press□1970□□

[24] Grattan-Guinness□*Search for Mathematical Roots*□471□footnote 28□

[25] William Ewald□ed.□*From Kant to Hilbert: A Source Book in the Foundations of Mathematics*□Oxford University Press□1996□□Vol. II□1113□

[26] □□ Paolo Mancosu□“Between Russell and Hilbert: Behmann on the Foundations of Mathematics”□*The Bulletin of Symbolic Logic*□Vol. 5□No. 3□Sept.1999□□321□

[27] Wilfried Sieg□“Hilbert’s Programs 1917-1922”□*The Bulletin of Symbolic Logic*□Vol. 5□No. 1□March 1999□□22□

[28] □□□□□□□□□□□□□*Grundzüge der Theoretischen Logik*□Verlag von Julius Springer□1928□□□2□□1938□□□□□□□□□□□□□□□□□□□□□□□2□□□□□□□Chelsea □□□□□1950□□□□□□□1□□□□□□□□

[29] □□□□□□□□□□□*Grundzüge der Theoretischen Logik*□68□

[30] *From Kant to Hilbert*□Vol. II□1157-1165□

[31] □□□□□ “Über formal unentscheidbare Sätze der Principia mathematica und verwandter Systeme I”□□□□*Monatshefte für Mathematik Und Physik*□Vol. 38□1931□□173-198□ □□□□□□□□□□□□□□Bernard Meltzer□□□□□□□Kurt Gödel□*On Formally Undecidable Propositions of Principia Mathematica and Related Systems*□Basic Books□1962□Dover Publications□1992□□□□□□□□□□□□□□□□□□□Elliott Mendelson□□□□□□□□Martin Davis□□ *The Undecidable: Basic Papers on Undecidable Propositions□Unsolvable Problems and Computable Functions*□Raven Press□1965□□5-38□□□□□□□□□Jean van Heijenoort□□□Gödel□□□□□□□□□□□□□□□□□□ *From Frege to Gödel*□□596-616□□□□Kurt Gödel□*Collected Works□Volume I□1929-1936*□Oxford University Press□1986□□144-195□□□□□□□□□□□□□□□□□□ “Gödel 1931”□

[32] □□□*Hilbert*□198□

[33] *The Autobiography of Bertrand Russell*□1872-1914□153□

[34] http://nobelprize.org/nobel_prizes/literature/laureates/1950□

[35] □□□*Hilbert*□205□

[36] □□□□□□□□□□□*Grundzüge der Theoretischen Logik*□77□

[37] □□□·□□□“A Note on the Entscheidungsproblem”□*The Journal of Symbolic Logic*□Vol. 1□No.1□March 1936□□41□

[38] □□·□□□“On Computable Numbers, with an Application to the Entscheidungsproblem”□*Proceedings of the London Mathematical Society*□2nd Series□Vol. 42□1936□□231□□□□□67□□□

[39] □□□262□□□□277□□□

[40] □□□*Hilbert*□209□

[illegible]4[illegible]
[illegible]

[illegible]·[illegible]10[illegible]·[illegible]·[illegible][1][illegible]“[illegible]”[illegible]

> [illegible][2]

20[illegible]“[illegible]”[illegible]·[illegible]200[illegible]·[illegible]·[illegible]·[illegible]1709—1751[illegible]1747[illegible]*L’Homme Machine*[illegible][3][illegible]·[illegible]

[illegible]·[illegible]·[illegible]1912[illegible]6[illegible]23[illegible]1907[illegible]1908[illegible]1911[illegible]

[illegible]1922[illegible][4]1926[illegible]60[illegible][5]

[illegible]“[illegible]——[illegible]”[6][illegible]

[illegible]1929[illegible]1928[illegible]·[illegible]·[illegible]

[illegible]·[illegible]1879[illegible]

[illegible]·[illegible]

[illegible]·[illegible]

[illegible]1929[illegible]12[illegible]·[illegible]·[illegible]·[illegible]·[illegible]

[illegible]“[illegible]”[7][illegible]·[illegible]

1930[illegible]12[illegible]G. H. [illegible][8][illegible]*A Course of Pure Mathematics*[illegible]15[illegible]1931[illegible]

[illegible]*Mathematische Grundlagen der Quantenmechanik*[illegible]·[illegible]·[illegible]20[illegible]20[illegible]·[illegible]·[illegible]20[illegible]30[illegible]·[illegible]1933[illegible]·[illegible]·[illegible]·[illegible]

[illegible]·[illegible]1935[illegible]·[illegible]·[illegible]”[illegible]”[illegible]Equivalence of Left and Right Almost Periodicity[illegible]1935[illegible]·[illegible]

[illegible]

[illegible]1933[illegible]·[illegible]1919[illegible]

> [illegible][9]

1935[illegible]”[illegible]”[illegible]·[illegible]·[illegible]·[illegible]
[illegible]1897—1984[illegible]M. H. A.[illegible]·[illegible]
[illegible]

[illegible]”[illegible]’[illegible]’[illegible]”[10][illegible]”[illegible]”[illegible]”[illegible]”[illegible]——[illegible]1935[illegible][11] [illegible]300[illegible]

[illegible]1936[illegible]4[illegible]”[illegible]”[12][illegible]

[illegible]1/3[illegible].010101…[illegible].001011011101111…[illegible]π[illegible]e[illegible]

[illegible]——[illegible]——[illegible]
[illegible]Description Number[illegible]——[illegible]——[illegible]π[illegible]e[illegible]

[illegible]”[illegible]”[illegible]”[illegible]”[illegible]

[illegible]

[illegible]

[illegible]0[illegible]0[illegible]”[illegible]”[illegible]”[illegible]”[illegible]262[illegible]263[illegible]

[illegible]·[illegible]·[illegible]”[illegible]”[13][illegible][14][illegible]

[illegible]·[illegible]1936[illegible]5[illegible]28[illegible]5[illegible]29[illegible]

> [illegible]·[illegible]·[illegible][15]

5[illegible]31[illegible]·[illegible]·[illegible]

> [illegible]”[illegible]”[illegible]Calculable Numbers[illegible]·[illegible]”[illegible]”[illegible]Computable Numbers[illegible][16]

[illegible]F. P. [illegible]

> [illegible]·[illegible]
>
> [illegible][17]

[illegible]Computability[illegible]“[illegible]”[illegible]1936[illegible]8[illegible]28[illegible]

[illegible]1936[illegible]11[illegible]12[illegible][18][illegible]1937[illegible]12[illegible][19][illegible]·[illegible]1937[illegible]5[illegible]*Journal of Symbolic Logic*[illegible]“[illegible]”[20][illegible]“[illegible]”[illegible]

[illegible]11[illegible]

[230]

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. TURING.

[Received 28 May, 1936. -- Read 12 November, 1936.]

The “computable" numbers may be described briefly as the real numbers whose expressions as a decimal are calculable by finite means.

[illegible]

A. M. Turing

[1936[illegible]5[illegible]28[illegible]1936[illegible]11[illegible]12[illegible]]

[illegible]

[illegible]

[illegible]“[illegible]”[illegible]1/3[illegible]0.33333…[illegible]π[illegible]3.14159…[illegible]“[illegible]”[illegible]1/3[illegible]π[illegible]“[illegible]”[illegible]“[illegible]4[illegible]7[illegible]0……”[illegible]1/3[illegible]π[illegible]

Although the subject of this paper is ostensibly the computable *numbers*, it is almost equally easy to define and investigate computabl functions of an integral variable or a real or computable variable, computable predicates, and so forth. The fundamental problems involved are, however, the same in each case, and I have chosen the computable numbers for explicit treatment as involving the least cumbrous technique. I hope shortly to give an account of the relations of the computable numbers, functions, and so forth to one another. This will include a development of the theory of functions of a real variable expressed in terms of computable numbers.

[illegible]

[illegible]I[illegible]

According to my definition, a number is computable if its decimal can be written down by a machine.

1936

·1791—1871“”Difference Engine1833“”Analytical Engine

Differential Analyzer1927·1890—1974MIT

2030·1910—19951935·1904—1995
19371937·1900—
1973IBMHarvard
Mark I[21]

··1897—1954·1909—1994[22]20
30

In §§9, 10 I give some arguments with the intention of showing that the computable numbers include all numbers which could naturally be regarded as computable. In particular, I show that certain large classes of numbers are computable. They include, for instance, the real parts of all algebraic numbers, the real parts of the zeros of the Bessel functions, the numbers *π*, *e*, etc.

§9§10πe

“”

[illegible]0[illegible]π[illegible]e[illegible]

[illegible]

The computable numbers do not, however, include all definable numbers, and an example is given of a definable number which is not computable.

[illegible]

[illegible]

[illegible]

Although the class of computable numbers is so great, and in many ways similar to the class of real numbers, it is nevertheless enumerable.

[illegible]

[illegible]

In §8 I examine certain arguments which would seem to prove the contrary. By the correct application of one of these arguments, conclusions are reached which are superficially similar to those of Gödel†.

† Gödel, “Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I” *Monatshefte Math. Phys.*, 38 (1931), 173-198.

[illegible]§8[illegible]†[illegible]

† [illegible]Gödel[illegible]“Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I”[illegible] *Monatshefte Math. Phys.*[illegible]38[illegible]1931[illegible]173-198[illegible]

[illegible]1962[illegible]

These results

[231]

have valuable applications. In particular, it is shown (§11) that the Hilbertian Entscheidungsproblem can have no solution.

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□§11□□

□□□□□18□□□□□□□□□□□□□

□□□□□□□□·□□

In a recent paper Alonzo Church† has introduced an idea of “effective calculability", which is equivalent to my “computability”, but is very differently defined. Church also reaches similar conclusions about the Entscheidungsrproblem‡. The proof of equivalence between “computability” and “effective calculability” is outlined in an appendix to the present paper.

† Alonzo Church, “An unsolvable problem of elementary number theory”, *American J. of Math.*, 58 (1936), 345-363.

‡ Alonzo Church, “A note on the Entscheidungsproblem”, *J. of Symbolic Logic*, 1 (1936), 40-41.

□□□·□□□□□□□□□†□□□“□□□□□□”□effective calculability□□□□□□□□□“□□□□”□computability□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□‡□□□“□□□□“□”□□□□□□“□□□□□□□□□□□□□□□□□□□□□

† □□Alonzo Church□“An unsolvable problem of elementary number theory”□*American J. of Math.*□58□1936□□345-363□

‡ □□Alonzo Church□“A note on the Entscheidungsproblem”□*J. of Symbolic Logic*□1□1936□□40-41□

□□□□□□□□□□□28□□□□□□□□□□□□□□□□□□□□□2□□□□□□□1889□□□□□□□□□□□□□□□“□□□□”□□□□□□□30□□□□□□□□□□“□□□□”□□□□□□□□·□□□□□□□□□□□□□□□□McGraw-Hill□□□□1958□□□□

□□□□□11□□□1□□□□□

1. *Computing machines.*

We have said that the computable numbers are those whose decimals are calculable by finite means. This requires rather more explicit definition. No real attempt will be made to justify the definitions given until we reach §9. For the present I shall only say that the justification

lies in the fact that the human memory is necessarily limited.

1.[illegible]

[illegible]§9[illegible]

[illegible]

[illegible]16[illegible]

[illegible]

We may compare a man in the process of computing a real number to a machine which is only capable of a finite number of conditions $q_1, q_2, \ldots, q_R$ which will be called "*m*-configurations".

[illegible]$q_1, q_2, q_3, \ldots, q_R$[illegible]"m-[illegible]"[illegible]

[illegible]m[illegible]machine[illegible]state[illegible]

The machine is supplied with a "tape" (the analogue of paper) running through it, and divided into sections (called "squares") each capable of bearing a "symbol".

[illegible]

[illegible]

[illegible]0[illegible]9[illegible]95[illegible]

[illegible]S[illegible]“symbol”[illegible]$\mathfrak{S}$[illegible]

At any moment there is just one square, say the *r*-th, bearing the symbol $\mathfrak{S}(r)$ which is “in the machine”.

[illegible]r[illegible]$\mathfrak{S}(r)$[illegible]“[illegible]”[illegible]

[illegible]$\mathfrak{S}(3451)$ [illegible]3451[illegible]“A”[illegible]$\mathfrak{S}(3451)$ [illegible]“A”[illegible]

[illegible]“[illegible]”[illegible]

We may call this square the “scanned square”. The symbol on the scanned square may be called the “scanned symbol”. The “scanned symbol” is the only one of which the machine is, so to speak, “directly aware”.

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

However, by altering its *m*-configuration the machine can effectively remember some of the symbols which it has “seen” (scanned) previously.

[illegible]m-[illegible]“[illegible]”[illegible]

[illegible]m-[illegible]m-[illegible]m-[illegible]q_{34}[illegible]“A”[illegible]m-[illegible]q_{17}[illegible]“B”[illegible]m-[illegible]q_{123}[illegible]m-[illegible]q_{17}[illegible]“[illegible]”[illegible]“A”[illegible]m-[illegible]q_{123}[illegible]“B”[illegible]q_{17}[illegible]q_{123}[illegible]q_{17}[illegible]q_{123}[illegible]

The possible behaviour of the machine at any moment is determined by the *m*-configuration q_n and the scanned symbol $\mathfrak{S}(r)$. This pair q_n, $\mathfrak{S}(r)$ will be called the “configuration” thus the configuration determines the possible behaviour of the machine.

[illegible]m-[illegible]q_n[illegible]$\mathfrak{S}(r)$[illegible]q_n[illegible]$\mathfrak{S}(r)$[illegible]“[illegible]”[illegible]

m-[illegible]q_1[illegible]q_2[illegible]m-[illegible]
[illegible]configuration[illegible]

[illegible]m-[illegible]

In some of the configurations in which the scanned square is blank (*i.e.* bears no symbol) the machine writes down a new symbol on the scanned square: in other configurations it erases the scanned symbol. The machine may also change the square which is being scanned, but only by shifting it one place to right or left.

[illegible]

[illegible]

In addition to any of these operations the *m*-configuration may be changed. Some of the symbols written down

[232]

will form the sequence of figures which is the decimal of the real number which is being computed. The others are just rough notes to “assist the memory”. It will only be these rough notes which will be liable to erasure.

[illegible]m-[illegible]“[illegible]”[illegible]

[illegible]

[illegible][23][illegible][24][illegible]“[illegible]”——[illegible]

It is my contention that these operations include all those which are used in the computation of a number.

[illegible]

[illegible]

The defence of this contention will be easier when the theory of the machines is familiar to the reader. In the next section I therefore proceed with the development of the theory and assume that it is understood what is meant by “machine”, “tape”, “scanned”, etc.

[illegible]

[illegible]

2. *Definitions.*

Automatic machines.

If at each stage the motion of a machine (in the sense of §1) is *completely* determined by the configuration, we shall call the machine an “automatic machine” (or *a*-machine.)

For some purposes we might use machines (choice machines or *c*-machines) whose motion is only partially determined by the configuration (hence the use of the word “possible” in §1).

2. [illegible]

[illegible]

[illegible]

[illegible]

[illegible]

When such a machine reaches one of these ambiguous configurations, it cannot go on until some arbitrary choice has been made by an external operator. This would be the case if we were using machines to deal with axiomatic systems. In this paper I deal only with automatic machines, and will therefore often omit the prefix *a*-.

[illegible]

[illegible]

[illegible]

Computing machines.

If an *a*-machine prints two kinds of symbols, of which the first kind (called figures) consists entirely of 0 and 1 (the others being called symbols of the second kind), then the machine will be called a computing machine.

[illegible]

[illegible] *a*-[illegible] 0 [illegible] 1 [illegible]

[illegible] 0 [illegible] 1 [illegible] [25] [illegible] 1937 [illegible] ·E. [illegible] 1916—2001 [illegible] 1937 [illegible] · [illegible] ENIAC [illegible] 1943—1945 [illegible] 1948 [illegible] [26] [illegible] bit [illegible] binary digit [illegible]

[illegible] 245 [illegible] 146 [illegible]

If the machine is supplied with a blank tape and set in motion, starting from the correct initial *m*-configuration, the subsequence of the symbols printed by it which are of the first kind will be called the *sequence computed by the machine.*

[illegible] m-[illegible] “[illegible]” [illegible]

[illegible] 0 [illegible] 1 [illegible] 0 [illegible] 1 [illegible]

The real number whose expression as a binary decimal is obtained by prefacing this sequence by a decimal point is called the *number computed by the machine.*

□□□□□□□□□□□□□□□□□□□□decimal point□□□□□□□□□□□□□□
□binary decimal□□□□□□□□□□□□□□□□□□

□□□
□□
□Windows□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

“□□□”□□□“decimal”□□□□□□“ten”□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□

.25

.5

.75

□□□□□□□decimal point□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□

.01

.1

.11

□□□□□□□□□□□□□□□□□□□□□□□□□□□binary point□□

□□□□□□□□□□□□□□□□□□□2□□□□□□□□□□□□□□□□□□□2□□□□

.1□□□2^{-1}□□□□□□□□□□□□□□1/2

.01□□□2^{-2}□□1/4

.001□2^{-3}□□1/8

.0001□2^{-4}□□1/16

.00001□2^{-5}□□1/32

□□□□□□□□□.10101□□□

$$1\times2^{-1}+0\times2^{-2}+1\times2^{-3}+0\times2^{-4}+1\times2^{-5}$$

□□□□□□□□□□□

$$\frac{1}{2}+\frac{0}{4}+\frac{1}{8}+\frac{0}{16}+\frac{1}{32}$$

□□□□□□□21/32□□.65625□□□□□□□□□□□□□□□□□□□□□□□□□□□□
1/3□□□□□□□

.01010101…

2/3□□

.10101010…

□□□□

1/5 □ .001100110011…

2/5 □ .011001100110…

3/5[illegible] .100110011001…

4/5 [illegible] .110011001100…

[illegible]“[illegible]decimal point[illegible]binary decimal[illegible]”

[illegible]“[illegible]”

[illegible]0[illegible]1[illegible]“[illegible]”[illegible]

0 1

[illegible]“[illegible]”[illegible]

.01

[illegible]1/4[illegible]

[illegible]0[illegible]1[illegible]

At any stage of the motion of the machine, the number of the scanned square, the complete sequence of all symbols on the tape, and the *m*-configuration will be said to describe the *complete configuration* at that stage.

[illegible]m-[illegible]

[illegible]“[illegible]”[illegible]

- ⟩ m-[illegible]
- ⟩ [illegible]m-[illegible]
- ⟩ [illegible]“[illegible]”[illegible]m-[illegible]

The changes of the machine and tape between successive complete configurations will be called the *moves* of the machine.

[illegible]“[illegible]”[illegible]

[illegible]

[233]

Circular and circle-free machines.

If a computing machine never writes down more than a finite number of symbols of the first kind, it will be called *circular.* Otherwise it is said to be *circle-free.*

A machine will be circular if it reaches a configuration from which there is no possible move, or if it goes on

moving, and possibly printing symbols of the second kind, but cannot print any more symbols of the first kind. The significance of the term “circular” will be explained in §8

[illegible]

[illegible] circular [illegible] circle-free [illegible]

[illegible] §8 [illegible]

[illegible] 0 [illegible] 1 [illegible]

[illegible] 0 [illegible] 1 [illegible] 1/4 [illegible] 0 [illegible] 1 [illegible] 0 [illegible]

[illegible]

Computable sequences and numbers.

A sequence is said to be computable if it can be computed by a circle-free machine. A number is computable if it differs by an integer from the number computed by a circle-free machine.

[illegible]

[illegible]

[illegible]

010000...

[illegible]

.010000...

[illegible]

1.010000...

[illegible]

10.01000...

[illegible]

We shall avoid confusion by speaking more often of computable sequences than of computable numbers.

[illegible]

[1] □□□·□□□*Alan Turing: The Enigma*□Simon & Schuster□1983□□11□□□□□□□□□□□□□□□□□

[2] □□□·□□□*Alan Turing*□13□

[3] Julien Offray de La Mettrie□*Machine Man and Other Writings*□□Ann Thomson□□□□□Cambridge University Press□1996□□

[4] □□□□*Alan Turing*□17□

[5] □□□□*Alan Turing*□21□

[6] □□□□*Alan Turing*□24□

[7] □□□□*Alan Turing*□46□

[8] □□□□□□□□□□□□□□□□2009□□□□——□□□

[9] □□□*Introduction to Mathematical Philosophy*□second edition□George Allen & Unwin Ltd□1920□Dover Publications□1993□□206□

[10] □□□□*Alan Turing*□93□

[11] □□□□*Alan Turing*□96□

[12] □□□□*Alan Turing*□109□

[13] □□□“A Note on the Entschiedungsproblem”□*The Journal of Symbolic Logic*□Vol. 1□No. 1□Mar.1936□□40-41□

[14] □□□“An Unsolvable Problem of Elementary Number Theory”□*American Journal of Mathematics*□Vol. 58□No. 2□Apr.1936□□345-363□□□□□□□□□□□□Martin Davis□ed.□*The Undecidable*□Raven Press□1965□□

[15] □□□□*Alan Turing*□113□

[16] □□□□*Alan Turing*□112□

[17] □□□□*Alan Turing*□113□

[18] □□□□□□□□□□□□□□□□□□□□11□□□□Vol. 42□3□□□November 30□1936□□□□□□□Vol. 42□4□□□dated December 23□1936□□1937□□□1936□10□□□1937□4□□□□□□□□□□□□□□□□□Vol. 42□□□□□□□□□□□□□□□□□□□□□1936□□□□□□□□□□□□□□1937□□Vol. 42□□□□□□□□1936~1937□□□□□□□□□Vol. 42□□□□□□

[19] □□□□Vol. 43□□7□□□December 30□1937□□□□□□Vol. 43□□□□□□□Vol. 43□□□□1937□5□□12□□□□□□□

[20] □□□“On Computable Numbers, with an Application to the Entscheidungsproblem”□□□□□*The Journal of Symbolic Logic*□Vol. 2□No. 1□Mar.1937□□42-43□

[21] [illegible]Paul E. Ceruzzi[illegible]*Reckoners*[illegible]*The Prehistory of the Digital Computer, from Relays to the Stored Program Concept*[illegible]1935-1945[illegible]Greenwood Press[illegible]1983[illegible]

[22] Robin Gandy[illegible]"The Confluence of Ideas in 1936"[illegible]in Rolf Herken[illegible]ed.[illegible]*The Universal Turing Machine: A Half-Century Survey*[illegible]Oxford University Press[illegible]1988[illegible]55-111[illegible]second edition[illegible]Springer-Verlag[illegible]1995[illegible]49-102[illegible]

[23] [illegible]Gregory J. Chaitin[illegible]"Computers, Paradoxes and the Foundations of Mathematics"[illegible]*American Scientist*[illegible]Vol. 90[illegible]March-April 2002[illegible]168[illegible]http://www.cs.auckland.ac.nz/CDMTCS/chaitin/amsci.pdf[illegible]

[24] *Soylent Green*[illegible]1973[illegible]

[25] [illegible]*Code*[illegible]*The Hidden Language of Computer Hardware and Software*[illegible]Microsoft Press[illegible]1999[illegible]

[26] [illegible]"A Mathematical Theory of Communication"*The Bell System Technical Journal*[illegible]Vol. 27[illegible]July[illegible]October 1948[illegible]J. W. Tukey[illegible]Lancelot Hogben[illegible] *The Vocabulary of Science*[illegible]Stein and Day[illegible]1970[illegible]146[illegible]"Tukey[illegible]bit[illegible]" [illegible]

[illegible]5[illegible]
[illegible]

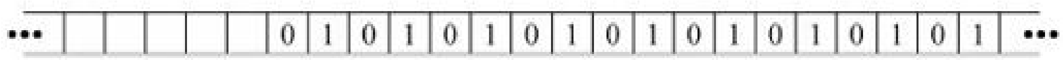
[illegible]

> 3. *Examples of computing machines.*
>
> I. A machine can be constructed to compute the sequence 010101….

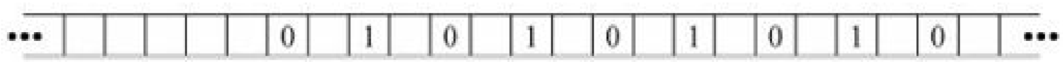
3. [illegible]

I. [illegible]010101…[illegible]

[illegible]



[illegible]



[illegible]m-[illegible]b[illegible]c[illegible]k[illegible]e[illegible]k[illegible]f[illegible]

> The machine is to have the four *m*-configurations “b”, “c”, “f”, “e” and is capable of printing “0” and “1”. The behaviour of the machine is described in the following table in which “*R*” means “the machine moves so that it scans the square immediately on the right of the one it was scanning previously”. Similarly for “*L*”. “*E*” means “the scanned symbol is erased” and “*P*” stands for “prints”.

[illegible]m-[illegible]“b”, “c”, “f”, “e”[illegible]0[illegible]1[illegible]“*R*”[illegible]“[illegible]”[illegible]“*L*”[illegible]“*E*”[illegible]“[illegible]”[illegible]“*P*”[illegible]“[illegible]”[illegible]

[illegible]*P*[illegible]*P*0[illegible]0[illegible]*P*1[illegible]1[illegible]*Px*[illegible]*x*[illegible]

> This table (and all succeeding tables of the same kind) is to be understood to mean that for a configuration described in the first two columns the operations in the third column are

carried out successively, and the machine then goes over into the *m*-configuration described in the last column.

[illegible]m-[illegible]

[illegible]

格局		行为	
m-格局	符号	操作	最终m-格局

[illegible]m-[illegible]*P*[illegible]*E*[illegible]*L*[illegible]*R*[illegible]m-[illegible]

[illegible]0[illegible]1[illegible]“Any”[illegible]“None”[illegible]“Any”[illegible]“[illegible]”[illegible]

When the second column is left blank, it is understood that the behaviour of the third and fourth columns applies for any symbol and for no symbol.

[illegible]

[illegible]

The machine starts in the *m*-configuration $\mathfrak{b}$ with a blank tape.

[illegible]m-[illegible]$\mathfrak{b}$[illegible]

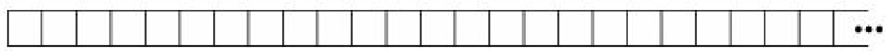
[illegible]m-[illegible]$\mathfrak{b}$[illegible]begin[illegible] begin [illegible]

Configuration		*Behaviour*	
m-config.	*symbol*	*operations*	*final m-config.*
$\mathfrak{b}$	None	$P0, R$	$\mathfrak{c}$
$\mathfrak{c}$	None	R	$\mathfrak{e}$
$\mathfrak{e}$	None	$P1, R$	$\mathfrak{f}$
$\mathfrak{f}$	None	R	$\mathfrak{b}$

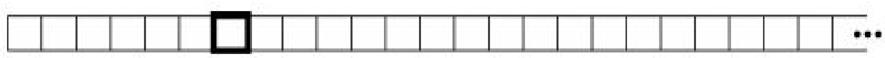
格局		行为	
m-格局	符号	操作	最终 m-格局
$\mathfrak{b}$	None	$P0, R$	$\mathfrak{c}$
$\mathfrak{c}$	None	R	$\mathfrak{e}$
$\mathfrak{e}$	None	$P1, R$	$\mathfrak{f}$
$\mathfrak{f}$	None	R	$\mathfrak{b}$

[illegible]m-[illegible] $\mathfrak{b}$ [illegible]“None”[illegible]0[illegible]m-[illegible] $\mathfrak{c}$ [illegible]

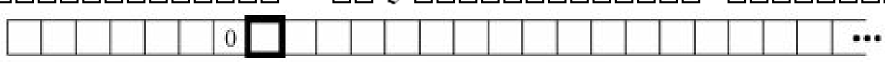
[illegible]m-[illegible] $\mathfrak{b}$ [illegible]



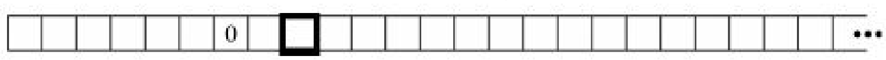
[illegible]



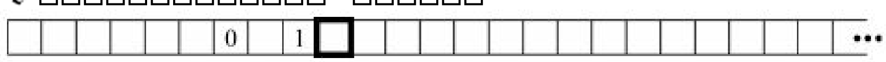
[illegible]m-[illegible] $\mathfrak{b}$ [illegible]0[illegible]



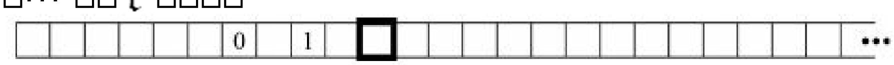
[illegible]m-[illegible] $\mathfrak{c}$ [illegible]m-[illegible] $\mathfrak{e}$ [illegible]



[illegible]m-[illegible] $\mathfrak{e}$ [illegible]1[illegible]



[illegible]m-[illegible] $\mathfrak{f}$ [illegible]



[illegible]m-[illegible] $\mathfrak{b}$ [illegible]0[illegible]1[illegible]

[illegible]“[illegible]”[illegible]

[illegible]0[illegible]1[illegible]——[illegible]

——[illegible]“[illegible]”[illegible]

[illegible]m-[illegible]“[illegible]”[illegible]·[illegible]35[illegible]Z1[illegible]Z3[illegible]“[illegible]”[illegible]

[illegible]“None”[illegible]“[illegible]”[illegible]

[illegible]

01010101…

[illegible]

.01010101…

[illegible]1/3[illegible]1[illegible]0[illegible]

.10101010…

[illegible]2/3[illegible]

[illegible]1/4[illegible]

.01000000…

[illegible]b[illegible]c[illegible]d[illegible]e[illegible]f[illegible]m-[illegible]

格局		行为	
m-格局	符号	操作	最终 m-格局
$\mathfrak{b}$	None	$P0, R$	$\mathfrak{c}$
$\mathfrak{c}$	None	R	$\mathfrak{d}$
$\mathfrak{d}$	None	$P1, R$	$\mathfrak{e}$
$\mathfrak{e}$	None	R	$\mathfrak{f}$
$\mathfrak{f}$	None	$P0, R$	$\mathfrak{e}$

[illegible]m-[illegible] $\mathfrak{e}$ [illegible] $\mathfrak{f}$ [illegible]0[illegible]0[illegible]“[illegible]”[illegible]

[illegible]

[illegible]1[illegible]“[illegible]”[illegible]

[234]

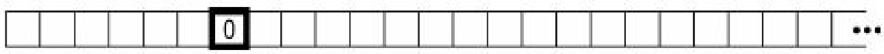
If (contrary to the description in §1) we allow the letters *L*, *R* to appear more than once in the operations column we can simplify the table considerably.

m-config.	symbol	operations	final m-config.
b	None	$P0$	b
	0	$R, R, P1$	b
	1	$R, R, P0$	b

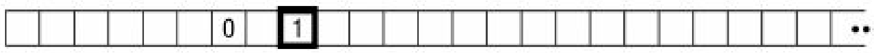
[illegible]§1[illegible]*L*[illegible]*R*[illegible]

m-格局	符号	操作	最终 m-格局
b	None	$P0$	b
	0	$R, R, P1$	b
	1	$R, R, P0$	b

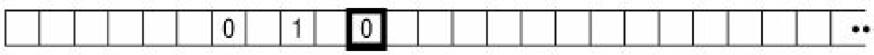
[illegible]*P*[illegible]m-[illegible]None[illegible]0[illegible]



[illegible]m-[illegible]0[illegible]1[illegible]



[illegible]1[illegible]0[illegible]



[illegible]0[illegible]1[illegible]

[illegible]“[illegible]”[illegible]

[illegible]*L*[illegible]*R*[illegible]*P*[illegible]

II. As a slightly more difficult example we can construct a machine to compute the sequence 001011011101111011111…

Ⅱ. [illegible]001011011101111011111…

“[illegible]”[illegible]1[illegible]0[illegible]1[illegible]1[illegible]1[illegible]

[illegible]1[illegible]“[illegible]”[illegible]1[illegible]

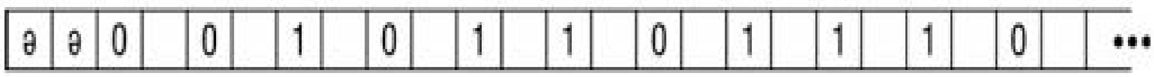
[illegible]m-[illegible]o[illegible]q[illegible]p[illegible]f[illegible]b[illegible]

The machine is to be capable of five *m*-configurations, viz. “o”, “q”, “p”, “f”, “b” and of printing “ə”, “*x*” “0” “1”. The

first three symbols on the tape will be “əə0”; the other figures follow on alternate squares.

□□□□□□5□m-□□□□“ 𝔬 ”, “ 𝔮 ”, “ 𝔭 ”, “ 𝔣 ”□“ 𝔟 ”, □□□□“ə”□“*x*”□“0”□“1”□□□□□□□□□□□□□□“əə0”□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□0□1□□□



□□□□□□□□□□□□“□□”□□□□□□□□□□□□□□□□□□□□□□□□□□□“□□□”□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□ə□□□sentinel□□□

□□□□□□□□□□0□1□□□

On the intermediate squares we never print anything but “*x*”. These letters serve to “keep the place” for us and are erased when we have finished with them.

□□□□□□□□□□□□□□□□□*x*□□□□□□□□□□□□□□□□□□□□□“□□□”□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□0□1□□□“□□”□□□“□□□”□□□0□1□□“□□□□”□□□□□□□□□□□□□□□□□□□□□□□□□F□□E□□□□□□□figures□erasable□□

We also arrange that in the sequence of figures on alternate squares there shall be no blanks.

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□0□1□□□·□□□□□□“□□□□□□□□”□[1]□□□□□□□“□□□”□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

Configuration		*Behaviour*	
m-config.	*symbol*	*operations*	*final m-config.*
𝔟		$P\text{ə}, R, P\text{ə}, R, P0, R, R, P0, L, L$	𝔬
𝔬	1	R, Px, L, L, L	𝔬
	0		𝔮
𝔮	Any (0 or 1)	R, R	𝔮
	None	$P1, L$	𝔭
𝔭	x	E, R	𝔮
	ə	R	𝔣
	None	L, L	𝔭
𝔣	Any	R, R	𝔣
	None	$P0, L, L$	𝔬

格局		行为	
m-格局	符号	操作	最终 m-格局
𝔟		$P_{\text{ə}}, R, P_{\text{ə}}, R, P0, R, R, P0, L, L$	𝔬
𝔬	1	R, Px, L, L, L	𝔬
	0		𝔮
𝔮	Any（0 or 1）	R, R	𝔮
	None	$P1, L$	𝔭
𝔭	x	E, R	𝔮
	ə	R	𝔣
	None	L, L	𝔭
𝔣	Any	R, R	𝔣
	None	$P0, L, L$	𝔬

□□□□□□□□□m-□□𝔟□□□□□□□□□□□□□□□□□0□□□□□□□

ə	ə	**0**		0																				…

m-□□𝔟□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□𝔟□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□□□□□□□□□□□□□□□

𝔮、𝔭□𝔣□□□□□□□□□□□□□□□□□□□*R*□*R*□□*L*□*L*□□□□□□□□□□□□□□□□□□

□□□𝔮□𝔣□□□□□□□□□□𝔭□□□□□□□

□□𝔟□□□□□m-□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□

m-□□𝔬□□□□□□□□□□1□□□□□□□□□□□□□□□□□□□1□□□□□□1□□□□□□□*x*□□□□□□□□□□□□□□□□□□□□□□□□m-□□𝔮□

m-□□𝔮□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□1□□□□□□□□□□□□□□□□m-□□𝔭□

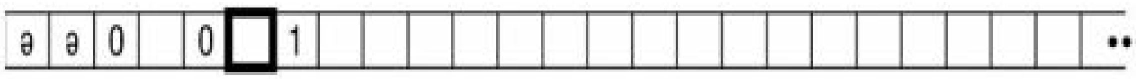
□□□□m-□□𝔣□□□□□□□□□□□□□□□□□□□□□□□□0□□□□□□□□□□□m-□□𝔬□

m-□□𝔭□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□*x*□□□□□□□*x*□□□□□□□□□□□□□□□m-□□𝔮□□□□□□□□□□□□□□□□□□□□□□m-□□𝔣□

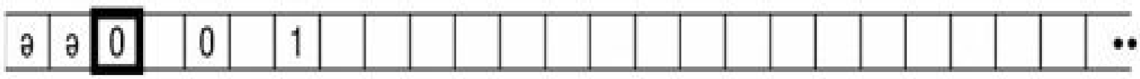
□□□□□□*x*□□□□□□□□□□□□□□□□□1□□□□□□□□□□□□1□□□□□1□□□□□*x*□□□□□□□□□□□□□□1□□□□□□□*x*□□□□□1□□□□□□□□□□1□

□□

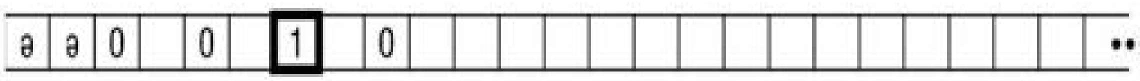
□□□m-□□𝔟□𝔬□□□□□□□□□0□𝔬□□□□□□□□□□□m-□□𝔮□□□m-□□𝔮□□□□□□0□1□□□□□□□□□□□□□□□m-□□□□□□□□□□□□□□□□□1□□□□□□□□□□m-□□𝔮□□□□□□□□□□□□□□□□□□□□□□□□1□□□□□□



□□□m-□□𝔭□□□□□□□□□□□□□□□□□□□□□□□□□*x*□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

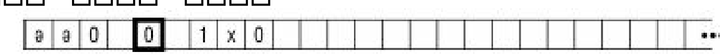


m-□□□𝔣□□□0□□□□□□□□□



□□□□□□□□1□□□□□□0□□□

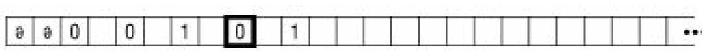
□□□□□□□m-□□𝔬□□□m-□□□□□□□□□□□1□□□□□□□1□□□□□□□□□□1□□□□□□*x*□□□□□□□□1□□□□0□□□□



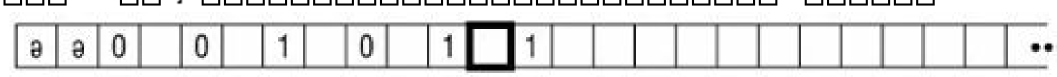
□□m-□□𝔮□□□m-□□□□□□□□□□□□□□□□□□□□□□1□□□□□□



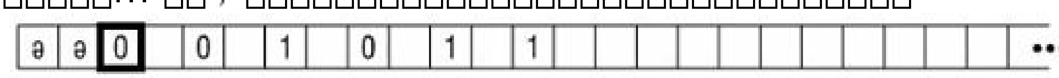
m-□□𝔭□□□□□□□□□□□□□□□□*x*□□□□□□□□□□□□*x*□□□□□□*x*□□□□□□



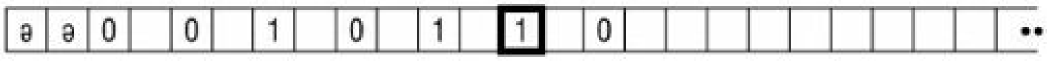
□□□□□m-□□𝔮□□□□□□□□□□□□□□□□□□□□□□□□□1□□□□□



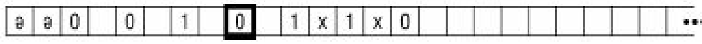
□□□□□□□m-□□𝔭□□□□□□□□□□□□□□□□□□□□□□□□□□□□



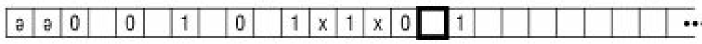
m-□□ 𝔣 □□□□□□□□□□□□□□□□□□□□□□□□□□0□□□□□□□□



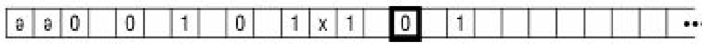
□□□□□□□□□□□□□□□□□□□□□1□□□□□□1□□□□□□□□□□□□□□□□□□□□□□□
m-□□ 𝔬 □□□□□□□□□□□□□1□□□□1□□□□□x□



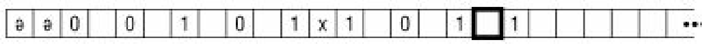
m-□□ 𝔮 □□□□□□□□□□□□□□□□□□□□□□□□□□□1□□□□□□



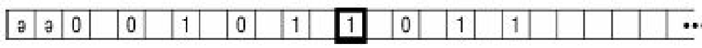
□□□□□□□□□□x□□□□□□□□□□□1□□□□1□□□□□□□□□□□x□□□□□□1□□□□
m-□□ 𝔭 □□□□□□□□□□□□□□□□□x□□□□□x□□□□□□



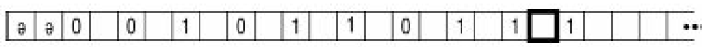
m-□□ 𝔮 □□□□□□□□□□□□□□□□□□□□□□□1□□□□□□



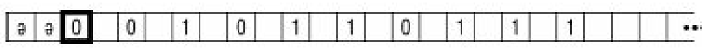
□□m-□□ 𝔭 □□□□□□□□□□□□x□□□x□□□□□□



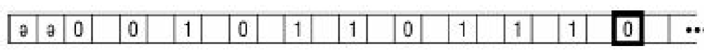
m-□□ 𝔮 □□□□□□□□□1□



□□□m-□□ 𝔭 □□□□□□□□□□□□□□□□□□□□



m-□□ 𝔣 □□□□□□□□□□□□□□□□□□□□□□0□



□□□□□□□□□□□□□□□□1□□□□□□□□□0□

□□x□

□□□“□□”□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□67□□□□□□□□“□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□□□□□□□□□□□□□□□□” □□□□□□□“□□□□□□□□”□□

□□□□□□□□□□□□□□□□——□□□□□□□□□□□□m-□□□□□——□□□□□□□□□□□□

> To illustrate the working of this machine a table is given below of the first few complete configurations. These complete configurations are described by writing down the sequence of symbols which are on the tape,
>
> [235]
>
> with the *m*-configuration written below the scanned symbol. The successive complete configurations are separated by colons.

□□m-□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

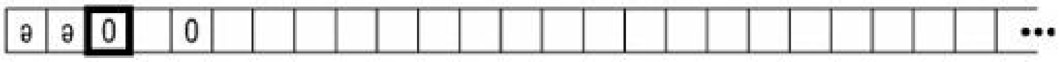
□□□□□□□□□□□□“□”□□

: ə ə 0   0 : ə ə 0   0 : ə ə 0   0 : ə ə 0   0   : ə ə 0   0 1 :
𝔟 𝔬 𝔮 𝔮 𝔮 𝔭

: ə ə 0   0 : ə ə 0   0 : ə ə 0   0 : ə ə 0   0   : ə ə 0   0 1 :
𝔟 𝔬 𝔮 𝔮 𝔮 𝔭

□□□□□□□□□□□□□□□□□□□□□□□□□□□0□0□□□□□□0□1□□m-□□

□□□ 𝔟 □□



□□m-□□□

: ə ə 0   0 :
𝔬

□□□□□□□0□□m-□□ 𝔬 □□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□□□□□□
𝔮 □

: ə ə 0   0 :
𝔮

□m-□□ 𝔮 □□□□□0□□□□□□□□□□□□□□□□m-□□□□□ 𝔮 □

: ə ə 0   0   :
𝔮

□□□□□□□□□0□□□□□□□□□□□□□ m-□□□□□ 𝔮 □

: ə ə 0   0   :
𝔮

□□□1□□□□□□□□□□□□□□ 𝔭 □

: ə ə 0   0   1 :
𝔭

□□□m-□□m-□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□m-□□ 𝔭 □□□□□□□□□□□□□□□□□□□□□□□□□ 𝔣 □□□□□□□□□□□

ə ə 0   0   1 : ə ə 0   0   1 : ə ə 0   0   1 : ə ə 0   0 1 :
𝔭 𝔭 𝔣 𝔣

$$\begin{array}{l} \text{əə}0\ \ 0\ \ 1{:}\text{əə}0\ \ 0\ \ 1{:}\text{əə}0\ \ 0\ \ 1{:}\text{əə}0\ \ 0\ \ 1{:} \\ \qquad\ \mathfrak{p}\qquad\quad \mathfrak{p}\qquad\qquad\quad \mathfrak{f}\qquad\qquad\qquad \mathfrak{f} \end{array}$$

[illegible]m-[illegible] $\mathfrak{f}$ [illegible]
0[illegible] $\mathfrak{o}$ [illegible]

$$\begin{array}{l} \text{əə}0\ \ 0\ \ \ 1{:}\text{əə}0\ \ \ 0\ \ 1\ \ \ \ {:}\text{əə}0\ \ \ 0\ \ 1\ \ 0{:} \\ \qquad\qquad\ \mathfrak{f}\qquad\qquad\qquad\qquad \mathfrak{f}\qquad\qquad\quad \mathfrak{o} \end{array}$$

$$\begin{array}{l} \text{əə}0\ \ 0\ \ \ 1{:}\text{əə}0\ \ \ 0\ \ 1\ \ \ \ {:}\text{əə}0\ \ \ 0\ \ 1\ \ 0{:} \\ \qquad\qquad\ \mathfrak{f}\qquad\qquad\qquad\qquad \mathfrak{f}\qquad\qquad\quad \mathfrak{o} \end{array}$$

m-[illegible] $\mathfrak{o}$ [illegible]1[illegible]*x*[illegible]

$$\begin{array}{l} \text{əə}0\ \ \ 0\ \ \ 1x0{:}\ \ldots\ldots \\ \qquad\quad\ \mathfrak{o} \end{array}$$

$$\begin{array}{l} \text{əə}0\ \ \ 0\ \ \ 1x0{:}\ \ldots\ldots \\ \qquad\quad\ \mathfrak{o} \end{array}$$

[illegible]

This table could also be written in the form

$$\mathfrak{b}{:}\text{əə}\mathfrak{o}0\ \ 0{:}\text{əə}\mathfrak{q}0\ \ 0{:}\ \ \ldots, \qquad\qquad \text{(C)}$$

in which a space has been made on the left of the scanned symbol and the *m*-configuration written in this space.

[illegible]

$$\mathfrak{b}{:}\text{əə}\mathfrak{o}0\ \ 0{:}\text{əə}\mathfrak{q}0\ \ 0{:}\ \ \ldots, \qquad\qquad \text{(C)}$$

[illegible]m-[illegible]

[illegible]C[illegible]Configuration[illegible]6[illegible]

$$\begin{array}{l} {:}\text{əə}0\ \ 0{:} \\ \qquad\ \mathfrak{o} \end{array}$$

[illegible]

$${:}\text{əə}\mathfrak{o}0\ \ 0{:}$$

[illegible]m-[illegible]m-[illegible]
[illegible]m-[illegible]

This form is less easy to follow, but we shall make use of it later for theoretical purposes.

[illegible]

[illegible]——[illegible]UTM[illegible]Universal Turing Machine[illegible]——[illegible]

[illegible]

[illegible]m-[illegible]m-[illegible]m-[illegible]“Any”[illegible]“None”[illegible]

[illegible]

The convention of writing the figures only on alternate squares is very useful: I shall always make use of it. I shall call the one sequence of alternate squares *F*-squares and the other sequence *E*-squares. The symbols on *E*-squares will be liable to erasure. The symbols on *F*-squares form a continuous sequence. There are no blanks until the end is reached.

[illegible]F-[illegible]E-[illegible]E-[illegible]F-[illegible]

[illegible]figures[illegible]0[illegible]1[illegible]erasable[illegible]“[illegible]”[illegible]F-[illegible]F-[illegible]F-[illegible]

E-[illegible]

There is no deed to have more than one *E*-square between each pair of *F*-squares: an apparent need of more *E*-squares can be satisfied by having a sufficiently rich variety of symbols capable of being printed of *E*-squares.

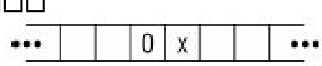
[illegible]F-[illegible]E-[illegible]E-[illegible]E-[illegible]

[illegible]E-[illegible]x[illegible]

If a symbol β is on an *F*-square S and a symbol α is on the *E*-square next on the right of S, then S and β will be said to be *marked* with α. The process of printing this α will be called marking β (or S) with α.

[illegible]β[illegible]F-[illegible]S[illegible]α[illegible]S[illegible]E-[illegible]S[illegible]β[illegible]α[illegible]α[illegible]α[illegible]β[illegible]S[illegible]

[illegible]0[illegible]F-[illegible]x[illegible]



[illegible]

[illegible]·[illegible][2][illegible]“[illegible]”[illegible]“[illegible]”[illegible]

(a) [illegible]

(b) [illegible]

(c) [illegible]

(d) [illegible]

(e) [illegible]

[illegible]

---

[1] □□□□·□□□□□□ “Recursive Unsolvability of a Problem of Thue”□*The Journal of Symbolic Logic*□Vol. 12□No. 1□Mar 1947□□1-11□□□□□□□□□□□□Martin Davis□ed.□*The Undecidable*□Raven Press□1965□□293-303□□□□□□B. Jack Copeland□ed.□*The Essential Turing*□Oxford University Press□2004□□97-101□

[2] □□□·□□□□“Finite Combinatory Processes Formulation I”□*The Journal of Symbolic Logic*□Vol. 1□No. 3□Sep. 1936□□103-105□□□□Martin Davis□ed.□*The Undecidable*□289-291□□□□□□□□□□□□□□□□□□□□□□□□□□□□□ *The Journal of Symbolic Logic* □□□1936□10□7□□□□□□ *the Proceedings of the London Mathematical Society* □1936□5□28□□□□□□□□□□□

[illegible]6[illegible]

[illegible]

[illegible]1935[illegible]5[illegible]·[illegible][1][illegible]“[illegible]”[2][illegible]

[illegible]·[illegible]52[illegible]

> [illegible][3]

[illegible]

[illegible]·[illegible]5[illegible]1929[illegible]

[illegible]1933[illegible]

[illegible]1936[illegible]9[illegible]·[illegible]

[illegible]·[illegible]·[illegible]G. H. [illegible]·[illegible]·[illegible]

[illegible]2000[illegible][4][illegible]1938[illegible]6[illegible]21[illegible]·[illegible]·[illegible]1500[illegible]

1939[illegible]·[illegible]·[illegible][5]

[illegible]1/3[illegible]0[illegible]1[illegible]1[illegible]3[illegible]

[illegible]

[illegible]

[illegible]——[illegible]1[illegible]

[illegible]“none”[illegible]“else”[illegible]**begin**[illegible]m-[illegible]

m-格局	符号	操作	最终 m-格局
begin	none	$P0$	**increment**
increment	0	$P1$	**rewind**
	1	$P0, L$	**increment**
	none	$P1$	**rewind**
rewind	none	L	**increment**
	else	R	**rewind**

m-[illegible]**begin**[illegible]0[illegible]**increment**[illegible]m-[illegible]**increment**[illegible]0[illegible]**increment**[illegible]1[illegible]1[illegible]**increment**[illegible]0[illegible]m-[illegible]**rewind**[illegible]

[illegible]

m-格局	符号	操作	最终 m-格局
begin	none	$P0$	**increment**
increment	0	$P1$	**rewind**
	1	$P2$	**rewind**
	2	$P3$	**rewind**
	3	$P4$	**rewind**
	4	$P5$	**rewind**
	5	$P6$	**rewind**
	6	$P7$	**rewind**
	7	$P8$	**rewind**
	8	$P9$	**rewind**
	9	$P0, L$	**increment**
	none	$P1$	**rewind**
rewind	none	L	**increment**
	else	R	**rewind**

[illegible]

+	**0**	**1**
0	0	1
1	1	10

和

×	**0**	**1**
0	0	0
1	0	1

[illegible]2[illegible]

$$\frac{\sqrt{2}}{2}$$

[illegible]0.70710678…[illegible]$\sqrt{2}$[illegible]

[illegible]$\sqrt{2}$[illegible]1.011[illegible]$1\frac{3}{8}$[illegible]1.375[illegible]1[illegible]1.0111[illegible]

```
  1.0111
 x1.0111
———————
  10111
  10111
  10111
  00000
```

10111

——————

10.00010001

□□□□□□□2□□□□□□□□□□□□□□□0□□□□□□□□□□□□1.0110□□□□□□□□□
□□□□□□□□□□□□1□□1.01101□□□□□

1.01101

x1.01101

——————

101101

101101

101101

000000

101101

——————

1.1111101001

□□□□□□□2□□□□□□□□□□□□□□□□□□□1.01101□□□□□□□□□$1\frac{3}{32}$□□
1.40625□

□□□□□□□□□□□□□□2□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□*n*□□□□□□□*m*□□□□□□□□□□□□□□(*n* × *m*)□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□*n*□□*m*□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□——□□□□□□□□□□□□□□□□□□□□□□□□□
□□□
□□□□□□□□□□□□□□□□

□□□□1.01101□□□□□□□□□□□□□□□□□□□□□□□□□□5□□□□□□□□36□□□
□□□□□□□□□□□□□□1□□1□□□□□1□□□□□□□□□□0□□□□□□□□□□□□□□□□□□
□□0
□□□□□□□□□□□□□□□□□□□□□□2□□□□□□□□□□□3□□□□□□5□□□□□□□□□□□
□□□

□□2□□□□□□□□□□□□□□□□□□□□□*n*□□□□□□□□□□□□□(2*n*-1)□□□□□□□□
□□□2□□□□□□□□1□□□□□□□□□□□2*n*□□□□□□□□□2□□□□□□□□□□□0□ □□□□
□□□□□□□□□□□□□□□□□□0□□1□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□F-□□□□□□2□□□□□□□□□□□□
□□□□□□——□□□□□□□□□□□□——□□□□□E-□□

□□□□□□□□m-□□**begin**□□□□□□@□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□1□□□

m-□□	□□	□□	□□ m-□□

begin	none	$P@, R, P1$	**new**

□□□□□□□□□□□□□□□□□2□□□□□1□□2□

□□□□□□□□□□□□□□□□□□□□m-□□**new**□□□□□□□□□□□□□□□□□□□□

new	@	R	**mark-digits**
	else	L	**new**

□□□1.25□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□“□□”□□□

@	1		0		1		?																	…

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□“□□”□□□□□□□□□□□□□□□□□□□□□□□x□□□□□□□□□□□□□□□□□□□□□II□□□75□□□□□□□m-□□**mark-digits**□□x□□□□□□□□□z□□r□

mark-digits	0	R, Px, R	**mark-digits**
	1	R, Px, R	**mark-digits**
	none	R, Pz, R, R, Pr	**find-x**

□□□□□□□□□□

@	1	x	0	x	1	x	?	z		r														…

r□□□□□□□□□□□□□□□□□□□□□0□□□□□□□□□□x□□□r□□□□□□□□□x□□□

find-x	x	E	**first-r**
	@	N	**find-digits**
	else	L, L	**find-x**
first-r	r	R, R	**last-r**
	else	R, R	**first-r**
last-r	r	R, R	**last-r**
	none	Pr, R, R, Pr	**find-x**

□□□□□□□□□□□□□□□□□□□□□□0000000□

@	1		0		1		?	z		r		r		r		r		r		r		r		…

□□□1□□0□

□□□□□□□□□□□□□□□□□□□□□□101□□□□□□1□□□□□□□□□□□□□□□1011□□□□□□□□□□□□□□□□□□□□□□□□□□□□x□y□z□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□x□□□□□□y□□□x□□□□□□□□□y□□□□□□□□□□□□□□x□y□□□□□□□□□□□□□□□□□□□□□z□□□□□□□□z□□□□□□□□□□□□

[illegible]1[illegible]*z*[illegible]*x*[illegible]*y*[illegible]*z*[illegible]

[illegible]*x*[illegible]*y*[illegible]*z*[illegible]m-[illegible]**find-digits**[illegible]**find-1st-digit**[illegible]*x*[illegible]*y*[illegible]*z*[illegible]

find - digits	@	R, R	**find - 1st - digit**
	else	L, L	**find - digits**
find - 1st - digit	x	L	**found - 1st - digit**
	y	L	**found - 1st - digit**
	z	L	**found - 2nd - digit**
	none	R, R	**find - 1st - digit**

[illegible]**find-1st-digit**[illegible]*x*[illegible]*y*[illegible]*z*[illegible]**found-1st-digit**[illegible]**found-2nd-digit**[illegible]

[illegible]0[illegible]0[illegible]**add-zero**[illegible]0[illegible]

found - 1st - digit	0	R	**add - zero**
	1	R, R, R	**find - 2nd - digit**

[illegible]1[illegible]*x*[illegible]*y*[illegible]

find - 2nd - digit	x	L	**found - 2nd - digit**
	y	L	**found - 2nd - digit**
	none	R, R	**find - 2nd - digit**

[illegible]

found - 2nd - digit	0	R	**add - zero**
	1	R	**add - one**
	none	R	**add - one**

[illegible]F-[illegible]1[illegible]*z*[illegible]**add-one**[illegible]1[illegible]

[illegible]0[illegible]

[illegible]E-[illegible]*r*[illegible]0[illegible]*s*[illegible]*t*[illegible]0[illegible]*u*[illegible]*v*[illegible]*w*[illegible]1[illegible]

m-[illegible]**add-zero**[illegible]*r*[illegible]*s*[illegible]*u*[illegible]*v*[illegible]

add - zero	r	Ps	**add - finished**
	u	Pv	**add - finished**
	else	R, R	**add - zero**

□*r*□□□0□□□*s*□□□0□□□□*u*□□□1□□□*v*□□□1□□□□□□□□□□□□□□□□
□□□□□□□

□1□□□□□□□□□□□□□*r*□□□0□□□*v*□□□1□□□□□□*u*□□□1□□□*s*□□□
0□□□□□□□□□□□□□□

$$\textbf{add-one}\begin{cases} r & Pv & \textbf{add-finished} \\ u & Ps, R, R & \textbf{carry} \\ \text{else} & R, R & \textbf{add-one} \end{cases}$$

□□□□□□□□□□□□□□□□□□□2□□□□□□□**new-digit-is-zero**□

$$\textbf{carry}\begin{cases} r & Pu & \textbf{add-finished} \\ \text{none} & Pu & \textbf{new-digit-is-zero} \\ u & Pr, R, R & \textbf{carry} \end{cases}$$

□□□□□□□□□□□□□□□□□□□□□□□□

@	1		0		1		?	z		v		r		r		r		r		r		r		•••

□□□□*r*□□□□*v*□□□1□□

□□□□□□*x*□*y*□*z*□□□□□□□□□□□□□□□*x*□□□□□□□□□□□□□*z*□□
□□*x*□*y*□□□□□□□□□□□*z*□□□*y*□□□□*x*□□□□□□□□□□□□□*x*□□□□□
□□□□□*y*□□□□□□□□*x*□□□□□□□□□□*y*□□□□□□□□□□□□

□□□□□□□□□□□

$$\textbf{add-finished}\begin{cases} @ & R, R & \textbf{erase-old-x} \\ \text{else} & L, L & \textbf{add-finished} \end{cases}$$

□□**erase-old-x**□□□*x*□□□□□□□□□□□□*z*□□□*y*□□□□□□□□□□
□□□□□□□□□□E-□□

$$\textbf{erase-old-x}\begin{cases} x & E, L, L & \textbf{print-new-x} \\ z & Py, L, L & \textbf{print-new-x} \\ \text{else} & R, R & \textbf{erase-old-x} \end{cases}$$

□□□□□□□□*x*□□□

$$\textbf{print-new-x}\begin{cases} @ & R, R & \textbf{erase-old-y} \\ y & Pz & \textbf{find-digits} \\ \text{none} & Px & \textbf{find-digits} \end{cases}$$

□□□□□□□□□□□□**find-digits**□□□□□□□□□□□□□□□

@	1		0		1	x	?	y		v		r		r		r		r		r		r		•••

□□□□□□□□□□□□□□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□*r*□□*u*□□

@	1		0		1	x	?	y		v		v		r		r		r		r		r		•••

□□□□□□□*x*□□□□□□□□□□

@	1		0	x	1		?	y		v		v		r		r		r		r		r		•••

□□□□□□□□□□□0□□□□□□□□□□□□□□□□□□□*r*□□□*s*□

@	1		0	x	1		?	y		v		v		s		r		r		r		r		•••

□□*x*□□□□□

@	1	x	0		1		?	y		v		v		s		r		r		r		r		•••

□□□□□□□□□□□□□□□□□□□*r*□□*v*□

@	1	x	0		1		?	y		v		v		s		v		r		r		r		•••

□□□*x*□□□□□□□□□□□□□□□□□□□□□□**erase-old-y**□**print-new-y**□□□

erase-old-y	y	$E,\ L,\ L$	**print-new-y**
	else	$R,\ R$	**erase-old-y**

print-new-y	@	R	**new-digit-is-one**
	else	$P\mathrm{y},\ R$	**reset-new-x**

□□□□□□□□□□□□□*y*□□1□

□□□□□□*x*□□□□□□□□□□□□□□□□□□□□□□□

reset-new-x	none	$R, P\mathrm{x}$	**flag-result-digits**
	else	R, R	**reset-new-x**

□□□□□□□□□□□*x*□*y*□□□□□□□□

@	1		0		1	y	?	x		v		v		s		v		r		r		r		•••

□□*s*□□*v*
□□□□□□□□□*t*□□*w*□

flag-result-digits	s	Pt, R, R	**unflag-result-digits**
	v	Pw, R, R	**unflag-result-digits**
	else	R, R	**flag-result-digits**

□□□*s*□*v*□□□□□□□*r*□*u*□

unflag-result-digits	s	Pr, R, R	**unflag-result-digits**
	v	Pu, R, R	**unflag-result-digits**
	else	N	**find-digits**

□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□*r*□□*u*□□□□

@	1		0		1	y	?	x		w		u		r		u		r		r		r		…

□□□2□□□□□□0□□□□□□**new-digit-is-zero**□□□□□*y*□□□□□□□□□□□□□□□□□□□□□□□□□□2□□□□**new-digit-is-one**□

□□□□□□□□□□□□□□□□□□□

new-digit-is-zero	@	R	**print-zero-digit**
	else	L	**new-digit-is-zero**

□□□□□□□□□□□□□□□0□□□□□□□□□□□□□□□□□□□□□

print-zero-digit	0	R, E, R	**print-zero-digit**
	1	R, E, R	**print-zero-digit**
	none	$P0, R, R, R$	**cleanup**

□□□m-□□**new-digit-is-one**□□□□1□□□□□□□□□□□**cleanup**□□□

State	Symbol	Operations	Final state
new-digit-is-one	@	R	**print-one-digit**
	else	L	**new-digit-is-one**
print-one-digit	0	R, E, R	**print-one-digit**
	1	R, E, R	**print-one-digit**
	none	$P1, R, R, R$	**cleanup**

[illegible]m-[illegible]**cleanup**[illegible]**new**[illegible]

State	Symbol	Operations	Final state
cleanup	none	N	**new**
	else	E, R, R	**cleanup**

[illegible]

@	1		0		1		1		?															•••

[illegible]sqrt[illegible]2[illegible]

[illegible]IEEE[illegible]52[illegible]15[illegible]16[illegible]

[illegible]

[illegible]

[1] □□□·□□□□*Alan Turing: The Enigma*□Simon & Schuster□1983□,95□

[2] □□□□*Alan Turing*□113□

[3] □□□

[4] □□·□□□“Systems of Logic Based on Ordinals”□*Proceedings of the London Mathematical Society*□2nd Series□Volume 45□1939□□161-228□□□□□·□□□*Collected Works of A.M.Turing: Mathematical Logic*□Elsevier□2001□□161-228□□□B.Jack Copeland□ed.□*The Essential Turing*□Oxford University Press□2004□□146-204□

[5] □□□□*Alan Turing*□152□

[illegible]7[illegible]

[illegible]

[illegible]2[illegible]m-[illegible]m-[illegible]

new	@	R	**mark-digits**
	else	L	**new**
new-digit-is-zero	@	R	**print-zero-digit**
	else	L	**new-digit-is-zero**
new-digit-is-one	@	R	**print-one-digit**
	else	L	**new-digit-is-one**

[illegible]m-[illegible]m-[illegible]

[illegible]m-[illegible]m-[illegible]

[illegible]m-[illegible]**goto-sentinel**[illegible]m-[illegible]**goto-sentinel**[illegible]

[illegible]**goto-sentinel**[illegible]

goto-sentinel	@	R	?????
	else	L	**goto-sentinel**

[illegible]m-[illegible]**goto-sentinel**[illegible]m-[illegible]m-[illegible]**goto-sentinel**[illegible]

[illegible]**goto-sentinel**[illegible]m-[illegible]

goto-sentinel(A)	@	R	**A**
	else	L	**goto-sentinel(A)**

[illegible]**new**[illegible]**new-digit-is-zero**[illegible]**new-digit-is-one**[illegible]m-[illegible]**begin**[illegible]**new**[illegible]**new**[illegible]**mark-digits**[illegible]

begin none *P*@, *R*, *P*1 **goto - sentinel(mark - digits)** □□□□□
□□□□**carry**□□□□□**new-digit-is-zero**□

$$\textbf{carry}\begin{cases} r & Pu & \textbf{add - finished} \\ \text{none} & Pu & \textbf{new - digit - is - zero} \\ u & \text{Pr}, R, R & \textbf{carry} \end{cases}$$

□□□□□□**goto-sentinel**□□□□□□**print-zero-digit**□

$$\textbf{carry}\begin{cases} r & Pu & \textbf{add - finished} \\ \text{none} & Pu & \textbf{goto - sentinel(print - zero - digit)} \\ u & Pr, R, R & \textbf{carry} \end{cases}$$

□□**print-zero-digit**□□□□□□□□**print-one-digit**□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□**print-digit**□□□□□□□□□□□□□□□□□□

$$\textbf{print - digit(a)}\begin{cases} 0 & R, E, R & \textbf{print - digit(a)} \\ 1 & R, E, R & \textbf{print - digit(a)} \\ \text{none} & Pa, R, R, R & \textbf{cleanup} \end{cases}$$

□□□□□□□□□*Pa*□□□□□□□□□□□□□□**print-digit**□□□□□□□□m-□□**carry**□□□□

$$\textbf{carry}\begin{cases} r & Pu & \textbf{add - finished} \\ \text{none} & Pu & \textbf{goto - sentinel(print - digit(0))} \\ u & Pr, R, R & \textbf{carry} \end{cases}$$

m-□□**print-new-y**□□□□□□□□□□□□□□□□□**new-digit-is-one**□□□□

$$\textbf{print - new - y}\begin{cases} @ & R & \textbf{goto - sentinel(print - digit(1))} \\ \text{else} & Py, R & \textbf{reset - new - x} \end{cases}$$

□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□
□□□□□□□□□□□□□□□□□□□□□□“□□”□□□□□□□□□□□□□“□□”□□□

□□□□□□“m-□□”□□□□□□□□□□□□□□□□□□□□□□□□□□“□□□”□□□□□□“□□□”□□□□
□□“□□□”□

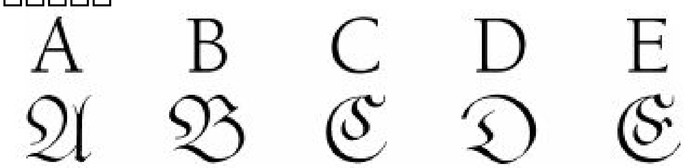
4. *Abbreviated tables.*

There are certain types of process used by nearly all machines, and these, in some machines, are used in many connections. These processes include copying down sequences of symbols, comparing sequences, erasing all symbols of a given form, etc. Where such processes are concerned we can abbreviate the tables for the *m*-configurations considerably by the use of “skeleton tables”. In skeleton tables there appear capital German letters and small Greek letters.

4. [illegible]

[illegible]“[illegible]”[illegible]m-[illegible]

[illegible]A[illegible]E[illegible]



[illegible]A[illegible]U[illegible]C[illegible]E[illegible]α[illegible]β[illegible]γ[illegible]

These are of the nature of “variables”. By replacing each capital German letter throughout by an *m*-configuration

[236]

and each small Greek letter by a symbol, we obtain the table for an *m*-configuration.

[illegible]“[illegible]”[illegible]m-[illegible]m-[illegible]

[illegible]m-[illegible]

[illegible]sqrt[illegible]Unix[illegible]Windows[illegible]Mac[illegible]

[illegible]

The skeleton tables are to be regarded as nothing but abbreviations: they are not essential. So long as the reader understands how to obtain the complete tables from the skeleton tables, there is no need to give any exact definitions in this connection.

[illegible]

[illegible]

[illegible] 7 [illegible] F- [illegible] E- [illegible] E- [illegible] F- [illegible] F- [illegible] F- [illegible]

Let us consider an example:

m-config.	*Symbol*	*Behaviour*	*Final m-config.*
$\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$	ə	L	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	not ə	L	$\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$
$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$	α		$\mathfrak{C}$
	not α	R	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	None	R	$\mathfrak{f}_2(\mathfrak{C}, \mathfrak{B}, \alpha)$
$\mathfrak{f}_2(\mathfrak{C}, \mathfrak{B}, \alpha)$	α		$\mathfrak{C}$
	not α	R	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	None	R	$\mathfrak{B}$

From the m-configuration $\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$ the machine finds the symbol of form α which is farthest to the left (the "first α") and the m-configuration then becomes $\mathfrak{C}$. If there is no α then the m-configuration becomes $\mathfrak{B}$.

[illegible]

m-格局	符号	行为	最终m-格局
$\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$	ə	L	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	not ə	L	$\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$
$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$	α		$\mathfrak{C}$
	not α	R	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	None	R	$\mathfrak{f}_2(\mathfrak{C}, \mathfrak{B}, \alpha)$
$\mathfrak{f}_2(\mathfrak{C}, \mathfrak{B}, \alpha)$	α		$\mathfrak{C}$
	not α	R	$\mathfrak{f}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$
	None	R	$\mathfrak{B}$

在 $\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$ 这个m-格局中，如果机器找到左边最远的α符号（第一个α），则m-格局转向$\mathfrak{C}$。如果没有α，则m-格局转向$\mathfrak{B}$。

[illegible]

[illegible]$\mathfrak{f}$[illegible]$\mathfrak{f}_1$[illegible]$\mathfrak{f}_2$[illegible]m-[illegible]$\mathfrak{f}$[illegible]m-[illegible]$\mathfrak{f}_1$[illegible]α[illegible]α[illegible]$\mathfrak{f}$[illegible]α[illegible]m-[illegible]$\mathfrak{C}$——$\mathfrak{f}$[illegible]m-[illegible]$\mathfrak{f}_1$[illegible]$\mathfrak{f}_2$[illegible]$\mathfrak{f}_1$[illegible]$\mathfrak{f}_2$[illegible]$\mathfrak{f}_1$[illegible]$\mathfrak{f}_2$[illegible]m-[illegible]$\mathfrak{B}$[illegible]$\mathfrak{B}$[illegible]$\mathfrak{f}$[illegible]α[illegible]

[illegible]$\mathfrak{f}$[illegible]find[illegible]α[illegible]m-[illegible]$\mathfrak{C}$[illegible]α[illegible]α[illegible]m-[illegible]$\mathfrak{C}$[illegible]

[illegible]m-[illegible]$\mathfrak{f}_1$[illegible]$\mathfrak{f}_2$[illegible]“not α”[illegible]“[illegible]α[illegible]”[illegible]“None”[illegible]m-[illegible]“None”[illegible]“None”[illegible]“not ə”[1][illegible]

[illegible]m-[illegible]

m-[illegible]	[illegible]	[illegible]	[illegible]m-[illegible]

$$\mathfrak{f}(\mathfrak{q}, \mathfrak{r}, x)$$

□□□m-□□$\mathfrak{q}$□$\mathfrak{r}$□□□□□□□□□□□□□□x□□□□□□□□□□□□□□

If we were to replace $\mathfrak{C}$ throughout by $\mathfrak{q}$ (say), $\mathfrak{B}$ by $\mathfrak{r}$, and *α* by *x*, we should have a complete table for the *m*-configuration $\mathfrak{f}(\mathfrak{q}, \mathfrak{r}, x)$.

□□□□□□$\mathfrak{C}$□$\mathfrak{q}$□□□$\mathfrak{B}$□$\mathfrak{r}$□□□*α*□*x*□□□□□□□□□□m-□□$\mathfrak{f}(\mathfrak{q}, \mathfrak{r}, x)$□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

m-格局	符号	操作	最终m-格局
$\mathfrak{f}$	ə	L	$\mathfrak{f}_1$
	not ə	L	$\mathfrak{f}$
$\mathfrak{f}_1$	x		$\mathfrak{q}$
	not x	R	$\mathfrak{f}_1$
	None	R	$\mathfrak{f}_2$
$\mathfrak{f}_2$	x		$\mathfrak{q}$
	not x	R	$\mathfrak{f}_1$
	None	R	$\mathfrak{r}$

□□□□□□□□□□□□□□□□□$\mathfrak{f}$□□□□m-□□$\mathfrak{f}$□$\mathfrak{f}_1$□$\mathfrak{f}_2$□□□□□□□□□□□□□□□□□□□□□□□□

$\mathfrak{f}$ is called an “*m*-configuration function” or “*m*-function”.

$\mathfrak{f}$□□□ “m-□□□□”□“m-□□”□

□□□□□“□□□”□□□

The only expressions which are admissible for substitution in an *m*-function are the *m*-configurations and symbols of the machine. These have to be enumerated more or less explicitly:

they may include expressions such as $\mathfrak{p}(\mathfrak{e}, x)$; indeed they must if there are any *m*-functions used at all.

m-[illegible]m-[illegible]$\mathfrak{p}(\mathfrak{e}, x)$[illegible]m-[illegible]

[illegible]$\mathfrak{p}$[illegible]m-[illegible]m-[illegible]$\mathfrak{p}$[illegible]m-[illegible]

[illegible]m-[illegible]m-[illegible]m-[illegible]m-[illegible]

If we did not insist on this explicit enumeration, but simply stated that the machine had certain *m*-configurations (enumerated) and all *m*-configurations obtainable by substitution of *m*-configurations in certain *m*-functions, we should usually get an infinity of *m*-configurations; *e.g.*, we might say that the machine was to have the *m*-configuration $\mathfrak{q}$ and all *m*-configurations obtainable by substituting an *m*-configuration for $\mathfrak{C}$ in $\mathfrak{p}(\mathfrak{C})$. Then it would have $\mathfrak{q}, \mathfrak{p}(\mathfrak{q}), \mathfrak{p}(\mathfrak{p}(\mathfrak{q})), \mathfrak{p}(\mathfrak{p}(\mathfrak{p}(\mathfrak{q}))),$ … as *m*-configurations.

[illegible]m-[illegible]m-[illegible]m-[illegible]m-[illegible]m-[illegible]m-[illegible]$\mathfrak{q}$[illegible]$\mathfrak{p}(\mathfrak{C})$[illegible]m-[illegible]$\mathfrak{C}$[illegible]m-[illegible]m-[illegible]$\mathfrak{q}, \mathfrak{p}(\mathfrak{q}), \mathfrak{p}(\mathfrak{p}(\mathfrak{q})), \mathfrak{p}(\mathfrak{p}(\mathfrak{p}(\mathfrak{q}))),$ …[illegible]

[illegible]m-[illegible]m-[illegible]

Our interpretation rule then is this. We are given the names of the *m*-configurations of the machine, mostly expressed in terms of *m*-functions.

[illegible]m-[illegible]m-[illegible]

[illegible]m-[illegible]

We are also given skeleton tables. All we want is the complete table for the *m*-configurations of the machine. This is obtained by repeated substitution in the skeleton tables.

[illegible]m-[illegible]

□□m-□□□□□□□□□□□□m-□□□□□

[237]

Further examples.

(In the explanations the symbol “rarr;” is used to signify “the machine goes into the *m*-configuration…”)

□□□□

□□□□□□□“→”□□□□“□□□□□□m-□□……”□

□□□□□□□□□□□□□□□□□□□□□□□□□“□□”□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□□□m-□□

□□□□□□□□□□□□□□□m-□□□□□□□m-□□□□□□□□□□□□

$\mathfrak{e}(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{f}\left(\mathfrak{e}_1(\mathfrak{C}, \mathfrak{B}, \alpha), \mathfrak{B}, \alpha\right)$	From $\mathfrak{e}(\mathfrak{C}, \mathfrak{B}, \alpha)$ the first α is erased and → $\mathfrak{C}$. If there is no α → $\mathfrak{B}$.
$\mathfrak{e}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$	E	$\mathfrak{C}$	

$\mathfrak{e}(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{f}\left(\mathfrak{e}_1(\mathfrak{C}, \mathfrak{B}, \alpha), \mathfrak{B}, \alpha\right)$	经过$\mathfrak{e}(\mathfrak{C}, \mathfrak{B}, \alpha)$，如果遇到第一个$\alpha$，则擦除它，并且→$\mathfrak{C}$；如果没有$\alpha$，则→$\mathfrak{B}$。
$\mathfrak{e}_1(\mathfrak{C}, \mathfrak{B}, \alpha)$	E	$\mathfrak{C}$	

$\mathfrak{e}$ □□“□□”□□□□□□□□□$\mathfrak{f}$□□□□□□□□□□□□□α□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathfrak{f}$□□□□□□□□□$\mathfrak{e}_1$□□□□□□□$\mathfrak{f}$□□□□□α□□□□□$\mathfrak{e}_1$□$\mathfrak{e}_1$□□□□□□□□□□□□□□□□□□□m-□□$\mathfrak{C}$□□□$\mathfrak{f}$□□□□α□□□□□□□m-□□$\mathfrak{B}$□

□□$\mathfrak{e}_1$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathfrak{e}_1(\mathfrak{C})$□

□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□□□□□□□□□□$\mathfrak{e}_1$□□□□$\mathfrak{f}$□□□□□□□□“□□□”□□□□□□□□□□□□□□□□□□□□□□□$\mathfrak{f}$□□□□□□□□□□□□□$\mathfrak{f}$□□□α□□□□□□□□□□

□□□□□$\mathfrak{e}$ □□□□□□□□□□□□□□□□□□□□□□□□□

$\mathfrak{e}(\mathfrak{B}, \alpha)$	$\mathfrak{e}(\mathfrak{e}(\mathfrak{B}, \alpha), \mathfrak{B}, \alpha)$	From $\mathfrak{e}(\mathfrak{B}, \alpha)$ all letters α are erased and → $\mathfrak{B}$.

$\mathfrak{e}(\mathfrak{B}, \alpha)$	$\mathfrak{e}(\mathfrak{e}(\mathfrak{B}, \alpha), \mathfrak{B}, \alpha)$	经过$\mathfrak{e}(\mathfrak{B}, \alpha)$，将擦除所有的$\alpha$，然后→$\mathfrak{B}$。

□□□
□□□□□□□□□□□□□□□□□□□□

□□□□□□ $\mathfrak{e}$ □□□□□□□□□□□ $\mathfrak{e}$ □□□□□□□□*α*□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□*α*□□□□□□□□□□□□□□□□□□□□□□□□*α*□□□□□□□□□□□□□□□□□□
□□□*α*□□□□

□□□

□□□□□□□□ $\mathfrak{e}$ □□□□□ $\mathfrak{e}$ □□□□□□□□□□□□□ $\mathfrak{e}$ □□□□□□□□□□□□□□□□□□□□□m-□□□
□□□□□□□

The last example seems somewhat more difficult to interpret than most. Let us suppose that in the list of *m*-configurations of some machine there appears $\mathfrak{e}(\mathfrak{b}, x)$ (= $\mathfrak{q}$, say).

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□□□□□ $\mathfrak{e}(\mathfrak{b}, x)$□□□□□ $\mathfrak{q}$ □□
m-□□ $\mathfrak{e}$ □□□□□□□□□□□m-□□□□□□□□□□□□□□□□□□□□□□□ $\mathfrak{e}(\mathfrak{b}, x)$□□□ $\mathfrak{b}$ □□□□□□□m-□□□□□□□□□□□ $\mathfrak{e}$ □$(\mathfrak{b}, x)$□□□□□□□m-□□□□□□□□□□□□□□ $\mathfrak{q}$ □□□□□□□□□□m-□□□□□□□□□□□ $\mathfrak{q}$ □□□□□□□m-□□□

□□□□□□□□□□□□□m-□□□□□□ $\mathfrak{e}(\mathfrak{b}, x)$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□

The table is

	$\mathfrak{e}(\mathfrak{b}, x)$	$\mathfrak{e}(\mathfrak{e}(\mathfrak{b}, x), \mathfrak{b}, x)$
or		
	$\mathfrak{q}$	$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$.

□□□□□

	$\mathfrak{e}(\mathfrak{b}, x)$	$\mathfrak{e}(\mathfrak{e}(\mathfrak{b}, x), \mathfrak{b}, x)$
□□		
	$\mathfrak{q}$	$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$.

□□□□□□□□□□□□□“□□□□……”□□□□□□□□□□□□□□□□□□ $\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$□□□
□□□□m-□□□□□□□□□□□□□□□□ $\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$□□□□□□□□□□□m-□□□

Or, in greater detail:

$\mathfrak{q}$		$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$
$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$		$\mathfrak{f}(\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b}, x), \mathfrak{b}, x)$
$\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b}, x)$	E	$\mathfrak{q}$.

□□□□□□□□□□□□

$\mathfrak{q}$		$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$
$\mathfrak{e}(\mathfrak{q}, \mathfrak{b}, x)$		$\mathfrak{f}(\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b}, x), \mathfrak{b}, x)$
$\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b}, x)$	E	$\mathfrak{q}$.

□□□□□□□□$\mathfrak{q}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathfrak{e}_1$□□$\mathfrak{q}$，$\mathfrak{q}$□□□□□□□□□□m-□□□□□□□□□□□m-□□□□□□□

In this we could replace $\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b},$ *x*) by $\mathfrak{q}'$ and then give the table for $\mathfrak{f}$ (with the right substitutions) ad eventually reach a table in which no *m*-functions appeared.

□□□□□□□$\mathfrak{q}'$□□$\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b},$ *x*)□□□□□$\mathfrak{f}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□□□

□□□□□□$\mathfrak{q}$□□□□□$\mathfrak{e}$ ($\mathfrak{b}$, *x*)□□□□□□□□□q'□□$\mathfrak{e}_1(\mathfrak{q}, \mathfrak{b},$ □*x*), □□□□□□□□□□□$\mathfrak{e}_1$□$\mathfrak{f}$□

□□□

□□$\mathfrak{pe}$□□“□□□□□”□□□□□□□□F-□□□□β□□□□□□

$\mathfrak{pe}(\mathfrak{C}, \beta)$	$\mathfrak{f}(\mathfrak{pe}_1(\mathfrak{C}, \beta), \mathfrak{C}, \partial)$	From $\mathfrak{pe}(\mathfrak{C}, \beta)$ the machine prints β at the end of the sequence of symbols and $\rightarrow \mathfrak{C}$.
$\mathfrak{pe}_1(\mathfrak{C}, \beta)$ Any R, R	$\mathfrak{pe}_1(\mathfrak{C}, \beta)$	
None $P\beta$	$\mathfrak{C}$	

$\mathfrak{pe}(\mathfrak{C}, \beta)$	$\mathfrak{f}(\mathfrak{pe}_1(\mathfrak{C}, \beta), \mathfrak{C}, \partial)$	经过 $\mathfrak{pe}(\mathfrak{C}, \beta)$，机器在符号序列的末尾打印$\beta$，然后$\rightarrow \mathfrak{C}$。
$\mathfrak{pe}_1(\mathfrak{C}, \beta)$ Any R, R	$\mathfrak{pe}_1(\mathfrak{C}, \beta)$	
None $P\beta$	$\mathfrak{C}$	

□□□□□□□□□□□□□□□□□□□□$\mathfrak{f}$□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathfrak{f}$□□□□□□□□□□□□□□□□□□□$\mathfrak{f}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□85□□□□□□□□□□□□□□□□$\mathfrak{pe}$□□□□□□□□□□□□□□□□□□□m-□□$\mathfrak{f}$□□□□□□□□□□□□□□□□□□□□□□□□□E-□□□□□□□

□□□□□□□□□□F-□□□□□□□□□□□□□$\mathfrak{pe}_1$ □□F-□□□□□□□□□□□□□□□□□□□□□□□β□□□□□□□□□□□□□□□□□□β□□0□1□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□ $\mathfrak{l}$ □□□□□□□ $\mathfrak{r}$ □□□□□□□□□□□□□□□□□□□□$\mathfrak{f}$□□□□□□□□□□□$\mathfrak{f}'$□$\mathfrak{f}''$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$\mathfrak{l}(\mathfrak{C})$	L	$\mathfrak{C}$	From $\mathfrak{f}'(\mathfrak{C}, \mathfrak{B}, \alpha)$ it does the same as for $\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$ but moves to the left before $\rightarrow \mathfrak{C}$.
$\mathfrak{r}(\mathfrak{C})$	R	$\mathfrak{C}$	
$\mathfrak{f}'(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{f}(\mathfrak{l}(\mathfrak{C}), \mathfrak{B}, \alpha)$	
$\mathfrak{f}''(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{f}(\mathfrak{r}(\mathfrak{C}), \mathfrak{B}, \alpha)$	

$\mathfrak{l}(\mathfrak{C})$	L	$\mathfrak{C}$	$\mathfrak{f}'(\mathfrak{C}, \mathfrak{B}, \alpha)$与$\mathfrak{f}(\mathfrak{C}, \mathfrak{B}, \alpha)$类似，不同的是它在$\rightarrow \mathfrak{C}$之前需要向左移动。
$\mathfrak{r}(\mathfrak{C})$	R	$\mathfrak{C}$	
$\mathfrak{f}'(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{f}(\mathfrak{l}(\mathfrak{C}), \mathfrak{B}, \alpha)$	
$\mathfrak{f}''(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{f}(\mathfrak{r}(\mathfrak{C}), \mathfrak{B}, \alpha)$	

□□□□□□□□□□$\mathfrak{fl}$□$\mathfrak{fr}$□□□$\mathfrak{f}'$□$\mathfrak{f}''$□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathfrak{c}$□□□□“□□”□□□α□□□□□□□□□□□$\mathfrak{c}$□□□□□□□□□□□□□F-□□□□□□□□□□□$\mathfrak{pe}$□□□□□□□□□□□□□□□□F-□□□

$\mathfrak{c}(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{f}'(\mathfrak{c}_1(\mathfrak{C}), \mathfrak{B}, \alpha)$	$\mathfrak{c}(\mathfrak{C}, \mathfrak{B}, \alpha)$. The machine writes at the end the first symbol marked α and $\rightarrow \mathfrak{C}$.
$\mathfrak{c}_1(\mathfrak{C})$	β	$\mathfrak{pe}(\mathfrak{C}, \beta)$	

$\mathfrak{c}(\mathfrak{C}, \mathfrak{B}, \alpha)$		$\mathfrak{f}'(\mathfrak{c}_1(\mathfrak{C}), \mathfrak{B}, \alpha)$	$\mathfrak{c}(\mathfrak{C}, \mathfrak{B}, \alpha)$，表示机器在尾部写下由$\alpha$标记的第一个符号，然后$\rightarrow \mathfrak{C}$。
$\mathfrak{c}_1(\mathfrak{C})$	β	$\mathfrak{pe}(\mathfrak{C}, \beta)$	

□□□□□□□□□□$\mathfrak{f}'$□□□□□α□□$\mathfrak{c}_1$□□□□□□□□□□□□□□□□□□□□□□$\mathfrak{pe}$□□□□□□□□□□□□

[238]

The last line stands for the totality of lines obtainable from it by replacing β by any symbol which may occur on the tape

of the machine concerned.

[illegible]β[illegible]

[illegible]$\mathfrak{c}$[illegible]0[illegible]1[illegible]$\mathfrak{c}_1$[illegible]

$$\mathfrak{c}_1(\mathfrak{E}) \quad \begin{cases} 0 & \mathfrak{pe}(\mathfrak{E}, 0) \\ 1 & \mathfrak{pe}(\mathfrak{E}, 1) \end{cases}$$

[illegible]$\mathfrak{ce}$[illegible]“[illegible]”[illegible]

$\mathfrak{ce}(\mathfrak{E}, \mathfrak{B}, \alpha)$	$\mathfrak{c}(\mathfrak{ce}(\mathfrak{E}, \mathfrak{B}, \alpha), \mathfrak{B}, \alpha)$	$\mathfrak{ce}(\mathfrak{B}, \alpha)$. The machine copies down in order at the end all symbols marked α and erases the letters α; $\rightarrow \mathfrak{B}$.
$\mathfrak{ce}(\mathfrak{B}, \alpha)$	$\mathfrak{ce}(\mathfrak{ce}(\mathfrak{B}, \alpha), \mathfrak{B}, \alpha)$	

$\mathfrak{ce}(\mathfrak{E}, \mathfrak{B}, \alpha)$	$\mathfrak{c}(\mathfrak{ce}(\mathfrak{E}, \mathfrak{B}, \alpha), \mathfrak{B}, \alpha)$	$\mathfrak{ce}(\mathfrak{B}, \alpha)$，表示机器从尾部按顺序复制所有用$\alpha$标记的符号，然后擦除这些$\alpha$，最后$\rightarrow \mathfrak{B}$。
$\mathfrak{ce}(\mathfrak{B}, \alpha)$	$\mathfrak{ce}(\mathfrak{ce}(\mathfrak{B}, \alpha), \mathfrak{B}, \alpha)$	

[illegible]$\mathfrak{ce}$[illegible]$\mathfrak{c}$[illegible]α[illegible]$\mathfrak{e}$[illegible]$\mathfrak{ce}$[illegible]α[illegible]F-[illegible]77[illegible]1[illegible]

[illegible]$\mathfrak{c}$[illegible]$\mathfrak{f}$[illegible]$\mathfrak{f}$[illegible]$\mathfrak{e}$[illegible]$\mathfrak{e}$[illegible]$\mathfrak{f}$[illegible]$\mathfrak{ce}$[illegible]“[illegible]”[2][illegible]

[illegible]10^6 ZHz[3][illegible]

[illegible]$\mathfrak{re}$[illegible]“[illegible]”[illegible]α[illegible]β[illegible]α[illegible]β[illegible]F-[illegible]α[illegible]β[illegible]

$\mathfrak{re}(\mathfrak{E}, \mathfrak{B}, \alpha, \beta)$		$\mathfrak{f}(\mathfrak{re}_1(\mathfrak{E}, \mathfrak{B}, \alpha, \beta), \mathfrak{B}, \alpha)$	$\mathfrak{re}(\mathfrak{E}, \mathfrak{B}, \alpha, \beta)$. The machine replaces the first α by β and $\rightarrow \mathfrak{E} \rightarrow \mathfrak{B}$ if there is no α.
$\mathfrak{re}_1(\mathfrak{E}, \mathfrak{B}, \alpha, \beta)$	$E, P\beta$	$\mathfrak{E}$	

$\mathfrak{re}(\mathfrak{C}, \mathfrak{B}, \alpha, \beta)$		$\mathfrak{f}(\mathfrak{re}_1(\mathfrak{C}, \mathfrak{B}, \alpha, \beta), \mathfrak{B}, \alpha)$	$\mathfrak{re}(\mathfrak{C}, \mathfrak{B}, \alpha, \beta)$，机器用$\beta$来替换第一个$\alpha$并→$\mathfrak{C}$，如果没有$\alpha$，则→$\mathfrak{B}$。
$\mathfrak{re}_1(\mathfrak{C}, \mathfrak{B}, \alpha, \beta)$	$E, P\beta$	$\mathfrak{C}$	

[illegible]β[illegible]α[illegible]

$\mathfrak{re}(\mathfrak{B}, \alpha, \beta)$	$\mathfrak{re}(\mathfrak{re}(\mathfrak{B}, \alpha, \beta), \mathfrak{B}, \alpha, \beta)$	$\mathfrak{re}(\mathfrak{B}, \alpha, \beta)$. The machine replaces all letters α by β; →$\mathfrak{B}$.

$\mathfrak{re}(\mathfrak{B}, \alpha, \beta)$	$\mathfrak{re}(\mathfrak{re}(\mathfrak{B}, \alpha, \beta), \mathfrak{B}, \alpha, \beta)$	$\mathfrak{re}(\mathfrak{B}, \alpha, \beta)$，机器用$\beta$替代所有的$\alpha$字符，然后→$\mathfrak{B}$。

[illegible]

[illegible] $\mathfrak{ce}$ [illegible]“[illegible]”[illegible]

$\mathfrak{cr}(\mathfrak{C}, \mathfrak{B}, \alpha)$	$\mathfrak{c}(\mathfrak{re}(\mathfrak{C}, \mathfrak{B}, \alpha, \alpha), \mathfrak{B}, \alpha)$	$\mathfrak{cr}(\mathfrak{B}, \alpha)$ differs from $\mathfrak{ce}(\mathfrak{B}, \alpha)$ only in that the letters α are not erased. The m-configuration $\mathfrak{cr}(\mathfrak{B}, \alpha)$ is taken up when no letters "α" are on the tape.
$\mathfrak{cr}(\mathfrak{B}, \alpha)$	$\mathfrak{cr}(\mathfrak{cr}(\mathfrak{B}, \alpha), \mathfrak{re}(\mathfrak{B}, \alpha, \alpha), \alpha)$	

$\mathfrak{cr}(\mathfrak{C}, \mathfrak{B}, \alpha)$	$\mathfrak{c}(\mathfrak{re}(\mathfrak{C}, \mathfrak{B}, \alpha, \alpha), \mathfrak{B}, \alpha)$	$\mathfrak{cr}(\mathfrak{B}, \alpha)$与$\mathfrak{ce}(\mathfrak{B}, \alpha)$的区别仅在于，前者的$\alpha$字母不被擦除。当纸带上没有字母$\alpha$时，m-格局$\mathfrak{cr}(\mathfrak{B}, \alpha)$就被转走。
$\mathfrak{cr}(\mathfrak{B}, \alpha)$	$\mathfrak{cr}(\mathfrak{cr}(\mathfrak{B}, \alpha), \mathfrak{re}(\mathfrak{B}, \alpha, \alpha), \alpha)$	

[illegible]

[illegible]“[illegible]”[illegible]$\mathfrak{cp}$[illegible]“[illegible]”[illegible] $\mathfrak{cpe}$ [illegible]“[illegible]”[illegible]m-[illegible]m-[illegible]$\mathfrak{C}$[illegible]1[illegible]

$\mathfrak{cp}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$		$\mathfrak{f}'(\mathfrak{cp}_1(\mathfrak{C}_1\ \mathfrak{A}, \beta), \mathfrak{f}(\mathfrak{A}, \mathfrak{E}, \beta), \alpha)$
$\mathfrak{cp}_1(\mathfrak{C}, \mathfrak{A}, \beta)$	γ	$\mathfrak{f}'(\mathfrak{cp}_2(\mathfrak{C}, \mathfrak{A}, \gamma), \mathfrak{A}, \beta)$
$\mathfrak{cp}_2(\mathfrak{C}, \mathfrak{A}, \gamma)$	γ	$\mathfrak{C}$
	not γ	$\mathfrak{A}$.

The first symbol marked *alpha;* and the first marked *β* are compared. If there is neither *α* nor *β*, → $\mathfrak{E}$. If there are both and the symbols are alike, → $\mathfrak{C}$. Otherwise → $\mathfrak{A}$.

$\mathfrak{cpe}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$	$\mathfrak{cp}(\mathfrak{e}(\mathfrak{e}(\mathfrak{C}, \mathfrak{C}, \beta), \mathfrak{C}, \alpha), \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$

$\mathfrak{cpe}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E},$ differs from $\mathfrak{cpe}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E},$ *α*, *β*) in that in the case when there is similarity the first *α* and *β* are erased.

$\mathfrak{cpe}(\mathfrak{A}, \mathfrak{E}, \alpha, \beta)$	$\mathfrak{cpe}(\mathfrak{cpe}(\mathfrak{A}, \mathfrak{E}, \alpha, \beta), \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$.

$\mathfrak{cpe}(\mathfrak{A}, \mathfrak{E},$ *α*, *β*). The sequence of symbols marked *α* is compared with the sequence marked *β* → $\mathfrak{C}$ if they are similar. Otherwise → $\mathfrak{A}$. Some of the symbols *α* and *β* are erased.

$\mathfrak{cp}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$		$\mathfrak{f}'(\mathfrak{cp}_1(\mathfrak{C}_1\ \mathfrak{A}, \beta), \mathfrak{f}(\mathfrak{A}, \mathfrak{E}, \beta), \alpha)$
$\mathfrak{cp}_1(\mathfrak{C}, \mathfrak{A}, \beta)$	γ	$\mathfrak{f}'(\mathfrak{cp}_2(\mathfrak{C}, \mathfrak{A}, \gamma), \mathfrak{A}, \beta)$
$\mathfrak{cp}_2(\mathfrak{C}, \mathfrak{A}, \gamma)$	γ	$\mathfrak{C}$
	not γ	$\mathfrak{A}$.

α *β* *α* *β* → $\mathfrak{C}$ *α* *β* → $\mathfrak{C}$ → $\mathfrak{A}$

$\mathfrak{cpe}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$	$\mathfrak{cp}(\mathfrak{e}(\mathfrak{e}(\mathfrak{C}, \mathfrak{C}, \beta), \mathfrak{C}, \alpha), \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$

$\mathfrak{cpe}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$ $\mathfrak{cpe}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta$ *α*, *β*) *α* *β*

$\mathfrak{cpe}(\mathfrak{A}, \mathfrak{E}, \alpha, \beta)$	$\mathfrak{cpe}(\mathfrak{cpe}(\mathfrak{A}, \mathfrak{E}, \alpha, \beta), \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$

$\mathfrak{cpe}(\mathfrak{C}, \mathfrak{A}, \mathfrak{E}, \alpha, \beta)$ [illegible] α [illegible] β [illegible] $\rightarrow \mathfrak{C}$ [illegible] $\rightarrow \mathfrak{A}$ [illegible] α [illegible] β [illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible] $\mathfrak{q}$ [illegible] m-[illegible] $\mathfrak{g}$ [illegible]

[illegible] $\mathfrak{g}$ [illegible] $\mathfrak{q}$ [illegible] $\mathfrak{f}$ [illegible] $\mathfrak{f}$ [illegible] $\mathfrak{g}$ [illegible] $\mathfrak{q}$ [illegible] $\mathfrak{g}$ [illegible]

[illegible] $\mathfrak{g}$ [illegible] $\mathfrak{g}$ [illegible] $\mathfrak{g}$ [illegible] α [illegible]

[239]

m-config.	symbol	operations	final m-config.	
$\mathfrak{q}(\mathfrak{C})$	Any	R	$\mathfrak{q}(\mathfrak{C})$	$\mathfrak{q}(\mathfrak{C}, \alpha)$. The machine finds the last symbol of form α. $\rightarrow \mathfrak{C}$.
	None	R	$\mathfrak{q}_1(\mathfrak{C})$	
$\mathfrak{q}_1(\mathfrak{C})$	Any	R	$\mathfrak{q}(\mathfrak{C})$	
	None		$\mathfrak{C}$	
$\mathfrak{q}(\mathfrak{C}, \alpha)$			$\mathfrak{q}(\mathfrak{q}_1(\mathfrak{C}, \alpha))$	
$\mathfrak{q}_1(\mathfrak{C}, \alpha)$	α		$\mathfrak{C}$	
	not α	L	$\mathfrak{q}_1(\mathfrak{C}, \alpha)$	

$\mathfrak{q}(\mathfrak{E})$	Any	R	$\mathfrak{q}(\mathfrak{E})$	$\mathfrak{q}(\mathfrak{E},\ \alpha)$，机器找到由$\alpha$标记的最后一个符号，然后→$\mathfrak{E}$。
	None	R	$\mathfrak{q}_1(\mathfrak{E})$	
$\mathfrak{q}_1(\mathfrak{E})$	Any	R	$\mathfrak{q}(\mathfrak{E})$	
	None		$\mathfrak{E}$	
$\mathfrak{q}(\mathfrak{E}, \alpha)$			$\mathfrak{q}(\mathfrak{q}_1(\mathfrak{E}, \alpha))$	
$\mathfrak{q}_1(\mathfrak{E}, \alpha)$	α		$\mathfrak{E}$	
	not α	L	$\mathfrak{q}_1(\mathfrak{E}, \alpha)$	

[illegible]

[illegible] $\mathfrak{pe}$ [illegible] F-[illegible] $\mathfrak{pe}$ 2 [illegible] F-[illegible]

$\mathfrak{pe}_2(\mathfrak{E}, \alpha, \beta)$	$\mathfrak{pe}(\mathfrak{pe}(\mathfrak{E}, \beta), \alpha)$	$\mathfrak{pe}_2(\mathfrak{E}, \alpha, \beta)$. The machine prints α β at the end.
$\mathfrak{pe}_2(\mathfrak{E}, \alpha, \beta)$	$\mathfrak{pe}(\mathfrak{pe}(\mathfrak{E}, \beta), \alpha)$	$\mathfrak{pe}_2(\mathfrak{E}, \alpha, \beta)$，机器在纸带尾部打印$\alpha$和$\beta$。

[illegible] $\mathfrak{ce}$ [illegible] α [illegible] $\mathfrak{ce}$ 2 [illegible] α [illegible] β [illegible] $\mathfrak{ce}$ 3 [illegible] α [illegible] β [illegible] γ [illegible]

$\mathfrak{ce}_2(\mathfrak{B}, \alpha, \beta)$	$\mathfrak{ce}(\mathfrak{ce}(\mathfrak{B}, \beta), \alpha)$	$\mathfrak{ce}_3(\mathfrak{B}, \alpha, \beta, \gamma)$. The machine copies down at the end first the symbols marked α, then those marked β, and finally those marked γ; it erases the symbols α, β, γ.
$\mathfrak{ce}_3(\mathfrak{B}, \alpha, \beta, \gamma)$	$\mathfrak{ce}(\mathfrak{ce}_2(\mathfrak{B}, \beta, \gamma), \alpha)$	

$\mathfrak{ce}_2(\mathfrak{B}, \alpha, \beta)$	$\mathfrak{ce}(\mathfrak{ce}(\mathfrak{B}, \beta), \alpha)$	
$\mathfrak{ce}_3(\mathfrak{B}, \alpha, \beta, \gamma)$	$\mathfrak{ce}(\mathfrak{ce}_2(\mathfrak{B}, \beta, \gamma), \alpha)$	$\mathfrak{ce}_3(\mathfrak{B}, \alpha, \beta, \gamma)$，机器在尾部首先复制用$\alpha$标记的字符，接着是用$\beta$标记的字符，最后复制$\gamma$标记的字符。之后，这些$\alpha$、$\beta$和$\gamma$标记被擦除。

□□□□□□□□□□□□□□□□α□□□□□□□□□□□β□□□□□□□□□□□□□□□□6□□□□□□$\mathfrak{ce}_5$□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□$\mathfrak{e}$□□□□□□□□

$\mathfrak{e}(\mathfrak{C})$	ə	R	$\mathfrak{e}_1(\mathfrak{C})$	From $\mathfrak{e}(\mathfrak{C})$ the marks are erased from all marked symbols. → $\mathfrak{C}$.
	Not ə	L	$\mathfrak{e}(\mathfrak{C})$	
$\mathfrak{e}_1(\mathfrak{C})$	Any	R, E, R	$\mathfrak{e}_1(\mathfrak{C})$	
	None		$\mathfrak{C}$	

$\mathfrak{e}(\mathfrak{C})$	ə	R	$\mathfrak{e}_1(\mathfrak{C})$	经过$\mathfrak{e}(\mathfrak{C})$，所有的标记将从被标记的字符上擦除，然后→$\mathfrak{C}$。
	Not ə	L	$\mathfrak{e}(\mathfrak{C})$	
$\mathfrak{e}_1(\mathfrak{C})$	Any	R, E, R	$\mathfrak{e}_1(\mathfrak{C})$	
	None		$\mathfrak{C}$	

□□□□□□□□□□□□□Pascal□□□□□□□□□·□□□1934—□□□□□□□□□□□□□□□□□□+□□□□□□□□□Prentice-Hall□1975□□□

---

[1] [illegible]·[illegible] “Recursive Unsolvability of a Problem of Thue”[illegible]*The Journal of Symbolic Logic*[illegible]Vol. 12[illegible]No. 1[illegible]Mar. 1947[illegible]1-11[illegible]Martin Davis[illegible]ed.[illegible]*The Undecidable*[illegible]Raven Press[illegible]1965[illegible]293-303[illegible]B. Jack Copeland[illegible]ed.[illegible]*The Essential Turing*[illegible]Oxford University Press[illegible]2004[illegible]97-101[illegible]

[2] tar-pit[illegible]——[illegible]

[3] [illegible]10^{21}Hz[illegible]——[illegible]

[illegible]8[illegible]

[illegible]

[illegible]0[illegible]1[illegible]

[illegible]20[illegible]30[illegible]

1937[illegible]·[illegible][1][illegible]

[illegible]

[illegible]·[illegible]1878—1929[illegible]Enigma[illegible]1918[illegible]1923[illegible][2]

[illegible]26[illegible]26[illegible]

[illegible]26[illegible]26[illegible]A[illegible]T[illegible]T[illegible]A[illegible]

[illegible]26[illegible]

[illegible]D[illegible]S[illegible]S[illegible]D[illegible]

[illegible]26[illegible]A[illegible]26[illegible]A[illegible]

26 A 17576 26

1932 [3] bomb 1938 · 1905—1980 “ ” [4] bomb [5]

GS&CS

1938 · GC&CC 1936 1939 GC&CC 50 [6]

1939 9 1 9 4 ·

8

1939 bomb bombe 1940 “ Bombe” 30 [7]

“ Bombe” crib kisses Banbury banburismus

1941 [8] ·

[9]

1941[illegible]·[illegible]·[illegible]·[illegible][10][illegible]

1942[illegible]11[illegible]·[illegible]1889—1976[illegible]·[illegible]·[illegible]“[illegible]”[illegible]1948[illegible]”[illegible]”[illegible]

[illegible]60[illegible]

[illegible]JPEG[illegible]MP3[illegible]

[illegible]

[illegible]

5. *Enumeration of computable sequences.*

A computable sequence γ is determined by a description of a machine which computes γ. Thus the sequence 001011011101111...is determined by the table on p. 234, and, in fact, any computable sequence is capable of being described in terms of such a table.

5. [illegible]

[illegible]γ[illegible]γ[illegible]001011011101111...[illegible]234[illegible]

[illegible]75[illegible]II[illegible]

It will be useful to put these tables into a kind of standard form.

[illegible]

[illegible]1[illegible]58[illegible]69[illegible]I[illegible]

In the first place let us suppose that the table is given in the same form as the first table, for example, I on p. 233. That is to say, that the entry in the operations column is

always of one of the forms *E*: *E*, *R*: *E*, *L*: *Pα*: *Pα*, *R*: *Pα*, *L*: *R*: *L*: or no entry at all.

[illegible]233[illegible]I[illegible]*E*: *E*[illegible]*R*: *E*[illegible]*L*: *Pα*: *Pα*[illegible]*R*: *Pα*[illegible]*L*: *R*: *L*[illegible]

[illegible]

The table can always be put into this form by introducing more *m*-configurations.

[illegible]m-[illegible]

[illegible]II[illegible]77[illegible] $\mathfrak{b}$ [illegible]

格局		行为	
m-格局	符号	操作	最终 m-格局
$\mathfrak{b}$		*P*ə, *R*, *P*ə, *R*, *P*0, *R*, *R*, *P*0, *L*, *L*	$\mathfrak{o}$

[illegible]c[illegible]d[illegible]e[illegible]g[illegible]h[illegible]f[illegible]

格局		行为	
m-格局	符号	操作	最终 m-格局
$\mathfrak{b}$		*P*ə, *R*	$\mathfrak{c}$
$\mathfrak{c}$		*P*ə, *R*	$\mathfrak{d}$
$\mathfrak{d}$		*P*0, *R*	$\mathfrak{e}$
$\mathfrak{e}$		*R*	$\mathfrak{g}$
$\mathfrak{g}$		*P*0, *L*	$\mathfrak{h}$
$\mathfrak{h}$		*L*	$\mathfrak{o}$

[illegible]

Now let us give numbers to the *m*-configurations, calling them q_1, ..., q_R, as in §1. The initial *m*-configuration is always to be called q_1.

[illegible]m-[illegible]§1[illegible]q_1[illegible]q_2[illegible]…[illegible]q_R[illegible]m-[illegible]q_1[illegible]

[illegible]237[illegible]m-[illegible]q_1[illegible]q_{237}[illegible]

[illegible]II[illegible]m-[illegible]q_1[illegible]q_6[illegible]m-[illegible] $\mathfrak{b}$ [illegible]q_1[illegible]

格局	行为

m-□□	□□	□□	□□m-□□
q_1		Pə, R	q_2
q_2		Pə, R	q_3
q_3		$P0$, R	q_4
q_4		R	q_5
q_5		$P0$, L	q_6
q_6		L	q_7

We also give numbers to the symbols S_1, ..., S_m [240]

and, in particular, blank=S_0, 0=S_1, 1=S_2.

□□□□□□S_1□...□S_m□□□□□□□□□S_0□0□S_1□1□S_2□

□□□□□1□□□□0□□□□□□2□□□□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□II□□□□□□□□□ə□x□□□□□□□□□□□□□□□□□□□□□□

S_0□□□□

S_1□□0□

S_2□□1□

S_3□□ə□

S_4□□x□

□□2□□□□□□□□□□S_0□S_{14}□□□□

□II□□□□□□□□□□□□□□□□□

□□		□□	
m-□□	□□	□□	□□m-□□
q_1		PS_3, R	q_2
q_2		PS_3, R	q_3
q_3		PS_1, R	q_4
q_4		R	q_5
q_5		PS_1, L	q_6
q_6		L	q_7

The lines of the table are now of form

m-config.	*Symbol*	*Operations*	*Final m-config.*	
q_i	S_j	PS_k, L	q_m	(N_1)
q_i	S_j	PS_k, R	q_m	(N_2)
q_i	S_j	PS_k	q_m	(N_3)

[illegible]

m-[illegible]	[illegible]	[illegible]	[illegible]m-[illegible]	
q_i	S_j	PS_k, L	q_m	(N_1)
q_i	S_j	PS_k, R	q_m	(N_2)
q_i	S_j	PS_k	q_m	(N_3)

[illegible]

[illegible] N_1 [illegible] N_2 [illegible] N_3 [illegible]

[illegible] S_0 [illegible] S_0 [illegible]

Lines such as

q_i S_j E, R q_m

are to be written as

q_i S_j PS_0, R q_m

[illegible]

q_i S_j E, R q_m

[illegible]

q_i S_j PS_0, R q_m

我们也可以采用另一种说明方式，将像图灵的表格中那样的行作为一种行

and lines such as

q_i S_j R q_m

to be written as

q_i S_j PS_j, R q_m

In this way we reduce each line of the table to a line of one of the forms (N_1), (N_2), (N_3).

并将像这样的行：

q_i S_j R q_m

写作如下：

q_i S_j PS_j, R q_m

通过这种方式，我们把表格的每一行都简化成了形式（N_1）、（N_2）、（N_3）之一的那种行了。

为了将例子II中的机器转换为标准形式，需要将其表中的每一行都简化成只进行一个操作的行，并且每个操作都有打印和移动两部分。这部分工作相当简单，下面的表就是例子II的标准形式的开头部分。这里有前六行：

格局			
m-格局	符号	操作	最终m-格局
q_1	S_0	PS_3, R	q_2
q_2	S_0	PS_3, R	q_3
q_3	S_0	PS_1, R	q_4
q_4	S_0	PS_0, R	q_5
q_5	S_0	PS_1, L	q_6
q_6	S_0	PS_0, L	q_7

这些行实际上是替换了例子II表格中的第一个m-格局。

$$\mathfrak{o}\ \begin{cases} 1 & R, Px, L, L, L \qquad \mathfrak{o} \\ 0 & \qquad\qquad\qquad\quad\ \ q \end{cases}$$

□□m-□□ o □□□□□q_7□□□□□□□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□q_8□q_9□q_{10}□m-□□□ q □□q_{11}□m-□□q_7□□□

□□ m-□□	□□	□□ □□	□□m-□□
q_7	S_2	PS_2, R	q_8
q_7	S_1	PS_1	q_{11}

□□□□□□□□□□□□□□□□□m-□□q_8□q_9□q_{10}□□□

q_8	S_0	PS_4, L	q_9
q_9	S_2	PS_2, L	q_{10}
q_{10}	S_0	PS_0, L	q_7

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□q_8□□□□□□□□□□□□□□□□□x□□□□□□□□□□□□□□□□□□□□□□□□□□q_7□□□□□1□□□□□□□□□□□□□□□□□□□□□□”□□”□”□”□□”□□”□□

□□□N_1□□□N_2□□□N_3□□□S_0□□□S_1□0□S_2□1□□□□□□□□S_3□S_4□□□0□1□□□□□□□□□□□□□□□□□

□□□□□□□□□m-□□□□□L□R□□□□□□□□□

From each line of form (N_1) let us form an expression $q_i S_j S_k L q_m$;

□□□□□N_1□□□□□□□□□□□$q_i S_j S_k L q_m$□

□□“□m-□□q_i□□□□□□□S_j□□□□□S_k□□□□□□□□□m-□□q_m”□□N_2□N_3□□□□□□□□

from each line of form (N_2) we form an expression $q_i S_j S_k R q_m$; and from each line of form (N_3) we form an expression $q_i S_j S_k N q_m$.

Let us write down all expressions so formed from the table for the machine and separate them by semi-colons. In this way we obtain a complete description of the machine.

In this description we shall replace q_i by the letter "*D*" followed by the letter "*A*" repeated *i* times, and S_j by "*D*" followed by "*C*" repeated *j* times.

This new description of the machine may be called the *standard description* (S.D). It is made up entirely from the letters *"A", "C", "D", "L", "R", "N",* and from ";".

If finally we replace *"A"* by "1", *"C"* by "2", *"D"* by "3", *"L"* by "4", *"R"* by "5", *"N"* by "6", and ";" by "7" we shall have a description of the machine in the form of an arabic numeral.

The integer represented by this numeral may be called a *description number* (D.N) of the machine. The D.N determine the S.D and the structure of the

[241]

machine uniquely. The machine whose D.N is *n* may be described as $\mathcal{M}(n)$.

[illegible]D.N[illegible]D.N[illegible]S.D[illegible]D.N[illegible]*n*[illegible]$\mathcal{M}(n)$[illegible]

[illegible]

To each computable sequence there corresponds at least one description number, while to no description number does there correspond more than one computable sequence.

[illegible]

[illegible]

[illegible]

The computable sequences and numbers are therefore enumerable.

[illegible]

[illegible]

Let us find a description number for the machine I of §3.

[illegible]§3[illegible]I[illegible]

[illegible]

格局		行为	
m-格局	符号	操作	最终 m-格局
$\mathfrak{b}$	None	$P0, R$	$\mathfrak{c}$
$\mathfrak{c}$	None	R	$\mathfrak{e}$
$\mathfrak{e}$	None	$P1, R$	$\mathfrak{f}$
$\mathfrak{f}$	None	R	$\mathfrak{b}$

When we rename the *m*-configurations its table becomes:

q_1	S_0	PS_1, R	q_2
q_2	S_0	PS_0, R	q_3
q_3	S_0	PS_2, R	q_4

q_4	S_0	PS_0,R	q_1

[illegible]m-[illegible]

q_1	S_0	PS_1,R	q_2
q_2	S_0	PS_0,R	q_3
q_3	S_0	PS_2,R	q_4
q_4	S_0	PS_0,R	q_1

[illegible]

Other tables could be obtained by adding irrelevant lines such as

q_1	S_1	PS_1,R	q_2

[illegible]

q_1	S_1	PS_1,R	q_2

[illegible]0[illegible]

Our first standard form would be

q_1 S_0 S_1 R q_2; q_2 S_0 S_0 R q_3; q_3 S_0 S_2 R q_4; q_4 S_0 S_0 R q_1;.

[illegible]

$q_1S_0S_1Rq_2$; $q_2S_0S_0Rq_3$; $q_3S_0S_2Rq_4$; $q_4S_0S_0Rq_1$;.

[illegible]q_i[illegible]D[illegible]i[illegible]1[illegible]A[illegible]S_j[illegible]D[illegible]j[illegible]0[illegible]C[illegible]

The standard description is

DADDCRDAA;DAADDRDAAA;

DAAADDCCRDAAAA;DAAAADDRDA;

[illegible]

DADDCRDAA;DAADDRDAAA;

DAAADDCCRDAAAA;DAAAADDRDA;

[illegible]D[illegible]D[illegible]

⟩ [illegible]D[illegible]A[illegible]A[illegible]

⟩ [illegible]D[illegible]A[illegible]D[illegible]0[illegible]C[illegible]D[illegible]DC[illegible]0[illegible]DCC[illegible]1[illegible]C[illegible]

[illegible]1[illegible]A[illegible]2[illegible]C[illegible]3[illegible]D[illegible]5[illegible]R[illegible]7[illegible]

A description number is

31332531173113353111731113322531111731111335317

and so is

31332531173113353111731113322531111731111335317313 23253117

[illegible]

31332531173113353111731113322531111731111335317[illegible]

[illegible]

31332531173113353111731113322531111731111335317313 23253117[illegible]

[illegible]3132 3253117[illegible]”[illegible]”[illegible]$q_1 S_1 S_1 R q_2$[illegible]1/3[illegible]

[illegible]q_{35}[illegible]35[illegible]q_{35}[illegible]

$DAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA$

[illegible]

31111111111111111111111111111111111

[illegible]

[illegible]π[illegible]π[illegible]

π=3.14159265358979323846264338327 95…

[illegible]π[illegible]π[illegible]π[illegible]32[illegible]π[illegible]

[illegible]

[illegible]——[illegible]——[illegible]

[illegible]·[illegible]”[illegible]”[11][illegible]

[illegible]MS Word 2003[illegible]WinWord.exe[illegible]12 047 560[illegible]9600[illegible]2900[illegible]WinWord.exe[illegible]$10^{29\,000\,000}$[illegible]50[illegible]50[illegible]11 000[illegible]googol[illegible]10[illegible]100[illegible]WinWord.exe[illegible]

[illegible]

> A number which is a description number of a circle-free machine will be called a *satisfactory* number. In §8 it is shown that there can be no general process for determining whether a given number is satisfactory or not.

> [illegible]§8[illegible]

[illegible]0[illegible]1[illegible]

□1□ □□□·□□□□*Alan Turing: The Enigma*□Simon & Schuster□1983□□138□

□2□ David Kahn□*Seizing the Enigma: The Race to Break the German U-Boat Codes□1939-1943*□Houghton-Mifflin□1991□□ch. 3□

□3□ □□□·□□□□□□“How Polish Mathematicians Deciphered the Enigma”□*Annals of the History of Computing*□Vol. 3□No. 3□July 1981□□213-234□□□□□ Elisabeth Rakus-Andersson□“The Polish Brains Behind the Breaking of the Enigma Code Before and During the Second World War” □□Christof Teuscher□ed.□*Alan Turing: Life and Legacy of a Great Thinker*□Springer□2004□□419-439□

□4□ □□□□□“How Polish Mathematicians Deciphered the Enigma”□226□

□5□ Kahn□*Seizing the Enigma*□73□

□6□ □□□□*Alan Turing*□148□

□7□ Stephen Budiansky□*Battle of Wits: The Complete Story of Codebreaking in World War II*□Free Press□2000□□155 □□□□□Jack Gray and Keith Thrower□*How the Turing Bombe Smashed the Enigma Code*□Speedwell□2001□□

□8□ □□□□*Alan Turing*□218-9□

□9□ I. J. Good□“Early Work on Computers at Bletchley”□*Annals of the History of Computing*□Vol. 1□No. 1□July 1979□□41□

□10□ □□□□*Alan Turing*□206□

□11□□□□□·□□□“How Real are Real Numbers?”□*International Journal of Bifurcation Chaos*□Vol. 16□2006□□1841-1848□□□□Gregory J Chaitin□Thinking About Gödel Turing: Essays on Complexity□1970-2007□World Scientific□2007□□267-280□

[illegible]9[illegible]
[illegible]

[illegible]

6. *The universal computing machine.*

It is possible to invent a single machine which can be used to compute any computable sequence. If this machine $\mathcal{U}$ is supplied with a tape on the beginning of which is written the S.D of some computing machine $\mathcal{M}$,

[242]

then $\mathcal{U}$ will compute the same sequence as $\mathcal{M}$. In this section I explain in outline the behaviour of the machine. The next section is devoted to giving the complete table for $\mathcal{U}$.

6. [illegible]

[illegible] $\mathcal{U}$ [illegible] $\mathcal{M}$ [illegible] $\mathcal{U}$ [illegible] $\mathcal{M}$ [illegible] $\mathcal{U}$ [illegible]

[illegible] $\mathcal{M}$ [illegible] $\mathcal{U}$ [illegible]

[illegible] 0[illegible]1 [illegible]

[illegible] $\mathcal{U}$ [illegible] $\mathcal{M}$ [illegible]&md;&md;[illegible]*A*[illegible]*C*[illegible]*D*[illegible]*L*[illegible]*N*[illegible]*R*[illegible] $\mathcal{M}$ [illegible] $\mathcal{U}$ [illegible] $\mathcal{M}$ [illegible]

[illegible] $\mathcal{U}$ [illegible] $\mathcal{M}$ [illegible] $\mathcal{U}$ [illegible] $\mathcal{M}$ [illegible] $\mathcal{M}$ [illegible] $\mathcal{U}$ [illegible] $\mathcal{M}$ [illegible] $\mathcal{M}$ [illegible] $\mathcal{M}$ [illegible] $\mathcal{M}$ [illegible]

[illegible] $\mathcal{U}$ [illegible] $\mathcal{M}$ [illegible]“[illegible]”[illegible] $\mathcal{U}$ [illegible]

[illegible]

Let us first suppose that we have a machine $\mathcal{M}'$ which will write down on the *F*-squares the successive complete configurations of $\mathcal{M}$.

These might be expressed in the same form as on p. 235, using the second description, (C), with all symbols on one line.



Or, better, we could transform this description (as in §5) by replacing each *m*-configuration by *"D"* followed by *"A"* repeated the appropriate number of times, and by replacing each symbol by *"D"* followed by *"C"* repeated the appropriate number of times. The numbers of letters *"A"* and *"C"* are to agree with the numbers chosen in §5, so that, in particular, "0" is replaced by *"DC"*, "1" by *"DCC"*, and the blanks by *"D".*

These substitutions are to be made after th complete configurations have been put together, as in (C). Difficulties arise if we do the substitution first.

In each complete configuration the blans would all have to be replaced by *"D"*, so that the complete configuration would

not be expressed as a finite sequence of symbols.

[illegible]*D*[illegible]

[illegible]*D*[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]*D*[illegible]

If in the description of the machine II of §3 we replace “𝔡” by *“DAA”*, “ə” by *“DCCC”*, “𝔮” by *“DAAA”*, then the sequence (C) becomes:

DA: *DCCCDCCCDAADCDDC*: *DCCCDCCCDAAADCDDC*:...(C_1)

(This is the sequence of symbols of *F*-squares.)

[illegible]§3[illegible]II[illegible]𝔡[illegible]*DAA*[illegible]ə[illegible]*DCCC*[illegible]𝔮[illegible]*DAAA*[illegible](C)[illegible]

DA[illegible]*DCCCDCCCDAADCDDC*[illegible]*DCCCDCCCDAAADCDDC*[illegible]...(C_1)

[illegible]F-[illegible]

[illegible]𝔟[illegible]*DA*[illegible]*D*[illegible]0[illegible]*DC*[illegible]

[illegible]$\mathcal{M}'$[illegible]$\mathcal{M}$[illegible]F-[illegible]0[illegible]1[illegible]E-[illegible]0[illegible]1[illegible]$\mathcal{M}'$[illegible]F-[illegible]$\mathcal{M}$[illegible]E-[illegible]

[illegible]*D*[illegible]

⟩ [illegible]*D*[illegible]*A*,[illegible]A[illegible]

⟩ [illegible]*D*[illegible]*A*[illegible]*D*[illegible]0[illegible]*C*[illegible]*D*[illegible]

[illegible]*DC* [illegible]0[illegible]*DCC* [illegible]1[illegible]*C*[illegible]

It is not difficult to see that if $\mathcal{M}$ can be constructed, then so can $\mathcal{M}'$. The manner of operation of $\mathcal{M}'$ could be made to depend on having the rules of operation (*i.e.*, the S.D) of $\mathcal{M}$ written somewhere within itself (*i.e.* within $\mathcal{M}'$); each step could be carried out by referring to these rules.

[illegible]$\mathcal{M}$[illegible]$\mathcal{M}'$[illegible]$\mathcal{M}$[illegible]$\mathcal{M}'$[illegible]$\mathcal{M}'$[illegible]

[illegible]$\mathcal{M}$[illegible]$\mathcal{M}'$[illegible]$\mathcal{M}'$[illegible]$\mathcal{M}'$[illegible]

We have only to regard the rules as being capable of being taken out and exchanged for others and we have something very akin to the universal machine.

[illegible]

[illegible] $\mathcal{U}$ [illegible] $\mathcal{M}$ [illegible] “[illegible]” [illegible] $\mathcal{U}$ [illegible] $\mathcal{U}$ [illegible]

[illegible] $\mathcal{U}$ [illegible] $\mathcal{U}$ [illegible] $\mathcal{M}$ [illegible] $\mathcal{M}$ [illegible] *D* [illegible] *A* [illegible]

DAAADCC

[illegible] m-[illegible] q_3 [illegible] 1 [illegible] $\mathcal{M}$ [illegible] $\mathcal{M}$ [illegible] $\mathcal{U}$ [illegible] $\mathcal{U}$ [illegible] —— [illegible] m-[illegible] $\mathcal{U}$ [illegible] m-[illegible]

[illegible] m-[illegible]

[illegible] $\mathcal{M}'$ [illegible] $\mathcal{U}$ [illegible] $\mathcal{M}'$ [illegible] $\mathcal{M}$ [illegible]

One thing is lacking: at present the machine $\mathcal{M}'$ prints no figures.

[illegible] $\mathcal{M}'$ [illegible]

[illegible] $\mathcal{M}'$ [illegible] $\mathcal{U}$ [illegible] F-[illegible] $\mathcal{M}$ [illegible] *A* [illegible] *C* [illegible] *D* [illegible] E-[illegible] 0 [illegible] 1 [illegible]

We may correct this by printing between each successive pair of complete configurations the figures which appear in the new configuration but not in the old. Then (C_1) becomes

DDA : 0 : 0 : *DCCCDCCCDAADCDDC* : *DCCC*... (C_2)

It is not altogether obvious that the *E*-squares leave enough room for the necessary “rough work”, but this is, in fact, the case.

[illegible] (C_1) [illegible]

DDA : 0 : 0 : *DCCCDCCCDAADCDDC* : *DCCC*... (C_2)

[illegible] E-[illegible] ”[illegible]” [illegible]

(C_2) [illegible] *D* [illegible] (C_2) [illegible] (C_1) [illegible] 0 [illegible]

[illegible] $\mathcal{M}'$ [illegible] $\mathcal{U}$ [illegible] $\mathcal{M}$ [illegible] 0 [illegible] 1 [illegible] $\mathcal{M}'$ [illegible] $\mathcal{M}$ [illegible]

[illegible]$\mathcal{M}'$ [illegible]$\mathcal{U}$[illegible]

[illegible]$\mathcal{M}'$[illegible]“[illegible]”[illegible]0[illegible]1[illegible]0[illegible]1[illegible]0[illegible]1[illegible]$\mathcal{M}'$[illegible]$\mathcal{M}$[illegible]0[illegible]1[illegible]$\mathcal{M}'$[illegible]0[illegible]1[illegible]

[illegible]

The sequences of letters between the colons in expressions such as (C_1) may be used as standard descriptions of the complete configurations. When the letters are replaced by figures, as in §5, we shall have a numerical

[243]

description of the complete configuration, which may be called its description number.

[illegible]C_1[illegible]§5 [illegible]

[illegible]$\mathcal{M}'$[illegible]$\mathcal{U}$[illegible]

[illegible]·[illegible][1][illegible]·[illegible][2]

[illegible]

7. *Detailed description of the universal machine.*

A table is given below of the behaviour of this universal machine. The *m*-configurations of which the machine is capable are all those occurring in the first and last columns of the table, together with all those which occur when we write out the unabbreviated tables of those which appear in the table in the form of *m*-functions. *E.g.*, $\mathfrak{e}(\mathfrak{anf})$ appears in the table and is an *m*-function.

7. [illegible]

[illegible]m-[illegible]m-[illegible]$\mathfrak{e}(\mathfrak{anf})$[illegible]m-[illegible]

m-[illegible]$\mathfrak{anf}$ [illegible]m-[illegible] $\mathfrak{e}$ ($\mathfrak{anf}$) [illegible] $\mathfrak{e}$ [illegible]239[illegible]113[illegible]

$\mathfrak{e}(\mathfrak{C})$	ə	R	$\mathfrak{e}_1(\mathfrak{C})$
	Not ə	L	$\mathfrak{e}(\mathfrak{C})$
$\mathfrak{e}_1(\mathfrak{C})$	Any	R, E, R	$\mathfrak{e}_1(\mathfrak{C})$
	None		$\mathfrak{C}$

[illegible]$\mathfrak{C}$ [illegible]$\mathfrak{anf}$ [illegible]

Its unabbreviated table is (see p. 239)

$\mathfrak{e}(\mathfrak{anf})$	ə	R	$\mathfrak{e}_1(\mathfrak{anf})$
	not ə	L	$\mathfrak{e}(\mathfrak{anf})$
$\mathfrak{e}_1(\mathfrak{anf})$	Any	R, E, R	$\mathfrak{e}_1(\mathfrak{anf})$
	None		$\mathfrak{anf}$

Consequently $\mathfrak{e}_1(\mathfrak{anf})$ is an *m*-configuration of $\mathfrak{U}$.

[illegible]239[illegible]

$\mathfrak{e}(\mathfrak{anf})$	ə	R	$\mathfrak{e}_1(\mathfrak{anf})$
	not ə	L	$\mathfrak{e}(\mathfrak{anf})$
$\mathfrak{e}_1(\mathfrak{anf})$	Any	R, E, R	$\mathfrak{e}_1(\mathfrak{anf})$
	None		$\mathfrak{anf}$

□□□$\mathfrak{e}_1$($\mathfrak{anf}$)□$\mathcal{U}$□□□m-□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

When $\mathcal{U}$ is ready to start work the tape running through it bears on it the symbol ə on an *F*-square and again ə on the next *E*-square; after this, on *F*-squares only, comes the S.D of the machine followed by a double colon “::” (a single symbol, on an *F*-square). The S.D consists of a number of instructions, separated by semi-colons.

□$\mathcal{U}$□□□□□□□□□□□□□□□□F-□□□□□ə□□□□E-□□□□□ə□□□□
□□□F-□□□□□□□□□□□□□□□“□□”□□□□□□□F-□□□□□□□□□□□
□□□□□□□□

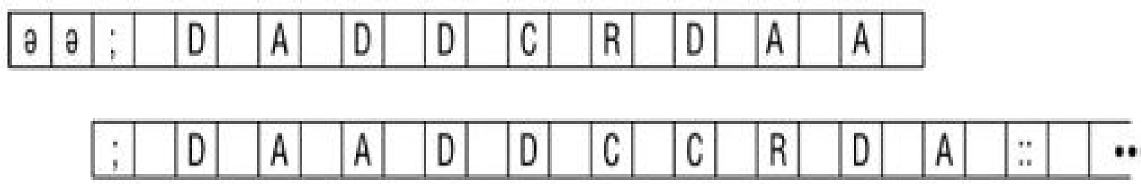
□□□□□□□□□□□□□□“□□”□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□

□□□□□□127□□□□5□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□“□□”□□□

□□□$\mathcal{U}$□□□□□□□□□□□□□□□$\mathcal{M}$□□□□□0□1□□□□□□□□□

m-□□	□□	□□	□□m-□□
q_1	S_0	PS_1, R	q_2
q_2	S_0	PS_2, R	q_1

□□□
□□□□□□□□□□□□□□□□□□□□□□□□□



□□□□$\mathcal{M}$□□□□$\mathcal{U}$□□□□$\mathcal{M}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

Each instruction consists of five consecutive parts

(i) *"D"* followed by a sequence of letters *"A"*. This describes the relevant *m*-configuration.

□□□□□□□□□□□□□□□

(i) *D*□□□□□*A*□□□□□□□□□□□□m-□□□

D□□□□□□□□*A*□□□□□□m-□□□□□□□□□□□□□□q_1□□□□q_0□

(ii) *"D"* followed by a sequence of letters *"C"*. This describes the scanned symbol.

(ii) *D*□□□□□*C*□□□□□□□□□□□□□

□□□□□□□□□*D*□□□□□□□□□*D*□□□□*C*□□0□□□*D*□□□□*C*□□1□

(iii) *"D"* followed by another sequence of letters *"C"*. This describes the symbol into which the scanned symbol is to be changed.

(iv) *"L"*, *"R"*, or *"N"*, describing whether the machine is to move to left, right, or not at all.

(v) *"D"* followed by a sequence of letters *"A"*. This describes the final *m*-configuration.

(iii) *D*□□□□□□□*C*□□□□□□□□□□□□□□□□□□□□□

(iv) *L*□*R*□*N*□□□□□□□□□□□□□□□□□□

(v) □□□*D*□□□□□*A*□□□□□□□□□□□m-□□□

□□□□□□□□□□□□□□□□□□□□□*A*□*C*□*D*□□□□□□□□□□□□□0□1□□□□□□□□□
□□□□□□□□□□□E-□□□□□

The machine $\mathcal{U}$ is to be capable of printing *"A"*, *"C"*, *"D"*, *"0"*, "1", *"u"*, *"v"*, *"w"*, *"x"*, *"y"*, *"z"*.

□□□□□$\mathcal{U}$□□□*A*□*C*□*D*□0□1□*u*□*v*□*w*□*x*□*y*□*z*□

□□□□□□"；"□□□□□□□□□□□□□□

The S.D is formed from ";", *"A"*, *"C"*, *"D"*, *"L"*, *"R"*, *"N"*.

S.D□;□*A*□*C*□*D*□*L*□*R*□*N*□□□

□□□□□□□□□□□□□□□□□□□□□□

[244]

Subsidiary skeleton table.

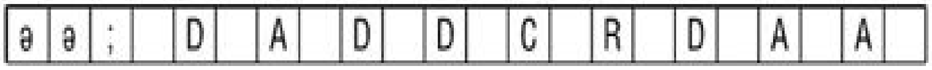
m-config.	symbol	operations	final m-config.	
$\mathrm{con}(\mathfrak{C}, \alpha)$	Not A	R, R	$\mathrm{con}(\mathfrak{C}, \alpha)$	$\mathrm{con}(\mathfrak{C}, \alpha)$. Starting from an F-square, S say, the sequence C of symbols describing a configuration closest on the right of S is marked out with letters α. $\rightarrow \mathfrak{C}$.
	A	$L, P\alpha, R$	$\mathrm{con}_1(\mathfrak{C}, \alpha)$	
$\mathrm{con}_1(\mathfrak{C}, \alpha)$	A	$R, P\alpha, R$	$\mathrm{con}_1(\mathfrak{C}, \alpha)$	
	D	$R, P\alpha, R$	$\mathrm{con}_2(\mathfrak{C}, \alpha)$	
$\mathrm{con}_2(\mathfrak{C}, \alpha)$	C	$R, P\alpha, R$	$\mathrm{con}_2(\mathfrak{C}, \alpha)$	$\mathrm{con}(\mathfrak{C}, \ )$. In the final configuration the machine is scanning the square which is four squares to the right of the last square of C. C is left unmarked.
	Not C	R, R	$\mathfrak{C}$	

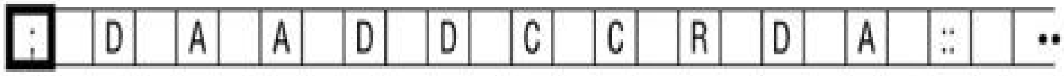
m-config.	symbol	operations	final m-config.	
$\mathrm{con}(\mathfrak{C}, \alpha)$	Not A	R, R	$\mathrm{con}(\mathfrak{C}, \alpha)$	$\mathrm{con}(\mathfrak{C}, \alpha)$，以一个F-格（比如$S$）开始，用$\alpha$标记描述距离$S$右边最近的那个格局的符号序列$C$。然后→$\mathfrak{C}$。
	A	$L, P\alpha, R$	$\mathrm{con}_1(\mathfrak{C}, \alpha)$	
$\mathrm{con}_1(\mathfrak{C}, \alpha)$	A	$R, P\alpha, R$	$\mathrm{con}_1(\mathfrak{C}, \alpha)$	
	D	$R, P\alpha, R$	$\mathrm{con}_2(\mathfrak{C}, \alpha)$	
$\mathrm{con}_2(\mathfrak{C}, \alpha)$	C	$R, P\alpha, R$	$\mathrm{con}_2(\mathfrak{C}, \alpha)$	$\mathrm{con}(\mathfrak{C}, \ )$，进入最终格局时，机器会扫描$C$的最后一格右侧的第四格，$C$没有被标记。
	Not C	R, R	$\mathfrak{C}$	

m- con configuration [3]

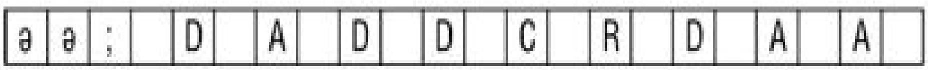
$$\mathrm{con}_1(\mathfrak{C}, \alpha) \quad \text{None} \quad PD,\ R,\ P\alpha,\ R,\ R,\ R \quad \mathfrak{C}$$

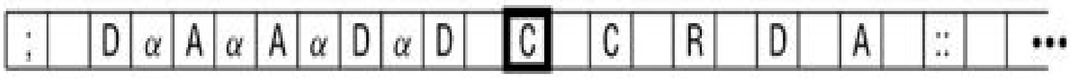
con





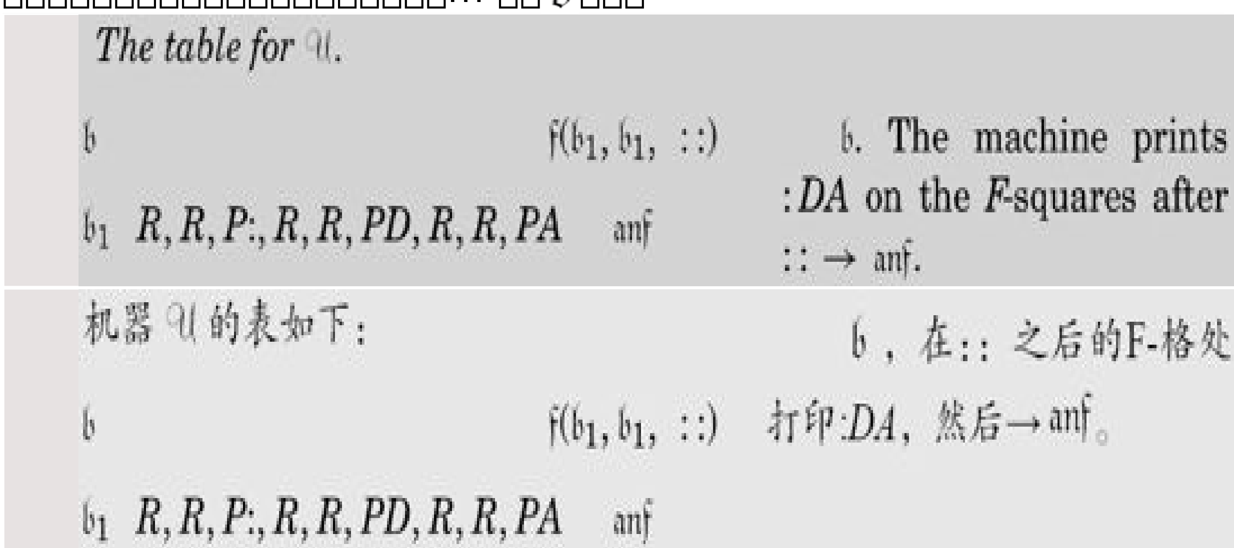
$\mathfrak{con}$ [illegible] A [illegible] A [illegible] α [illegible] $\mathfrak{con}_1$ [illegible] A [illegible] D [illegible] D [illegible] $\mathfrak{con}_2$ [illegible] $\mathfrak{con}_2$ [illegible] C [illegible] C [illegible] C [illegible]





[illegible] $\mathfrak{con}$ [illegible] C [illegible] “[illegible]” [illegible] “C [illegible]” [illegible] “[illegible]” [illegible] $\mathfrak{con}$ [illegible]

[illegible] m-[illegible] $\mathfrak{U}$ [illegible] m-[illegible] m-[illegible] $\mathfrak{b}$ [illegible]



The table for $\mathfrak{U}$.

$\mathfrak{b}$		$\mathfrak{f}(\mathfrak{b}_1, \mathfrak{b}_1, ::)$	$\mathfrak{b}$. The machine prints $:DA$ on the F-squares after $:: \rightarrow \mathfrak{anf}$.
$\mathfrak{b}_1$	$R, R, P:, R, R, PD, R, R, PA$	$\mathfrak{anf}$	

机器 $\mathfrak{U}$ 的表如下：

$\mathfrak{b}$		$\mathfrak{f}(\mathfrak{b}_1, \mathfrak{b}_1, ::)$	$\mathfrak{b}$，在:: 之后的F-格处打印$:DA$，然后$\rightarrow \mathfrak{anf}$。
$\mathfrak{b}_1$	$R, R, P:, R, R, PD, R, R, PA$	$\mathfrak{anf}$	

m-[illegible] $\mathfrak{f}$ [illegible] m-[illegible] m-[illegible] q_1 [illegible] DA [illegible] $\mathfrak{b}_1$ [illegible]

ə	ə	;		D		A		D		D		C		R		D		A		A	

;		D		A		A		D		D		C		C		R		D		A		::	

:		D		A		…

𝔘[illegible]m-[illegible] anf [illegible]·[illegible] *anfang*，*anfang* [illegible]“[illegible]”[illegible] 𝔤 [illegible] 𝔮 [illegible]

anf	g(anf$_1$, :)	anf. The machine marks the configuration in the last complete configuration with y. → fom.
anf$_1$	con(fom, y)	
anf	g(anf$_1$, :)	anf，机器用y标记最后一个完全格局中的格局，然后→fom。
anf$_1$	con(fom, y)	

𝔤 [illegible] con [illegible] *y* [illegible] m-[illegible] con$_1$ [illegible] *D* [illegible]

ə	ə	;		D		A		D		D		C		R		D		A		A	

;		D		A		A		D		D		C		C		R		D		A		::	

:		D	y	A	y	D	y	…

[illegible] m-[illegible] con [illegible] *D* [illegible] con$_1$ [illegible] *D* [illegible]

[illegible] *y* [illegible] m-[illegible] fom [illegible] fom [illegible] kom [illegible]“[illegible]”[illegible]

fom	;	R, Pz, L	con(fmp, x)
	z	L, L	fom
	not z nor ;	L	fom

fom. The machine finds the last semi-colon not marked with z. It marks this semi-colon with z and the configuration following it with x.

fom	;	R, Pz, L	con(fmp, x)
	z	L, L	fom
	not z nor ;	L	fom

fom，机器寻找最后一个没有被z标记的分号。它把这个分号标记上z，并且把该分号之后的格局标记为x。

□□□□□□ fom □□□□□□□□□□□□□□□□□□□□□□□□□□□□z□□□□ con □□□□□□□□□

ə	ə	;		D		A		D		D		C		R		D		A		A	

;	z	D	x	A	x	A	x	D	x	D		C		C		R		D		A		::	

:		D	y	A	y	D	y	…

□□z□□□□□□□□□□□□□□□□□□□□□□□□□□□□□ fom □□□□□□□□□□□□□z□□□□□□□

m-□□ fmp □□□□□□□□□□□□□“□□”□□□□□□□□ cpe □□□□□□x□□□□□□□□□□□m-□□□□□□
□□□□□□□□y□□□□□□□□□□□□□□□□□m-□□□□□□□□□□

fmp	cpe(e(fom, x, y), sim, x, y)

fmp. The machine compares the sequences marked x and y. It erases all letters x and y. → sim if they are alike. Otherwise → fom.

$\mathfrak{kmp}$	$\mathfrak{cpe}\left(\mathfrak{e}(\mathfrak{kom}, x, y), \mathfrak{sim}, x, y\right)$	$\mathfrak{kmp}$，机器比较标记为x和y的序列。它擦除所有的x和y。如果这些序列相似就→ $\mathfrak{sim}$ ，否则→ $\mathfrak{kom}$。

□□$\mathfrak{cpe}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□x□y□□□□□□
□□□$\mathfrak{sim}$□□□□□□□

□□□□□x□y□□□□□□□□□□□□□□□□□□□□□$\mathfrak{cpe}$□□□□□□□□□□□□$\mathfrak{e}$□□□□□
□□□□□□□x□y□□□□□□□□$\mathfrak{kom}$□□□□□□□□□

□□$\mathfrak{kmp}$□□□□□□□□□□□□□□□□□□□m-□□□□□□□□□□□□□□$\mathfrak{e}$□□□□□□□□□
□□□□y□□□□□□$\mathfrak{cpe}$□□□□□□□□□$\mathfrak{kom}$□□□□□□□□□□□□$\mathfrak{anf}$□□□□□□□□□□□□
·□□□□□□□□□□□□□

$$\mathfrak{kmp} \qquad \mathfrak{cpe}\left(\mathfrak{e}(\mathfrak{e}(\mathfrak{anf}, x), y), \mathfrak{sim}, x, y\right)$$

□□□□□□□□$\mathfrak{anf}_1$□□□□□□□□□□□m-□□□□□□□$\mathfrak{kom}$□□□□□□□□□□□□□□□□
□□□□□

ə	ə	;	z	D	x	A	x	D	x	D		C		R		D		A		A	

;	z	D		A		A		D		D		C		C		R		D		A		::	

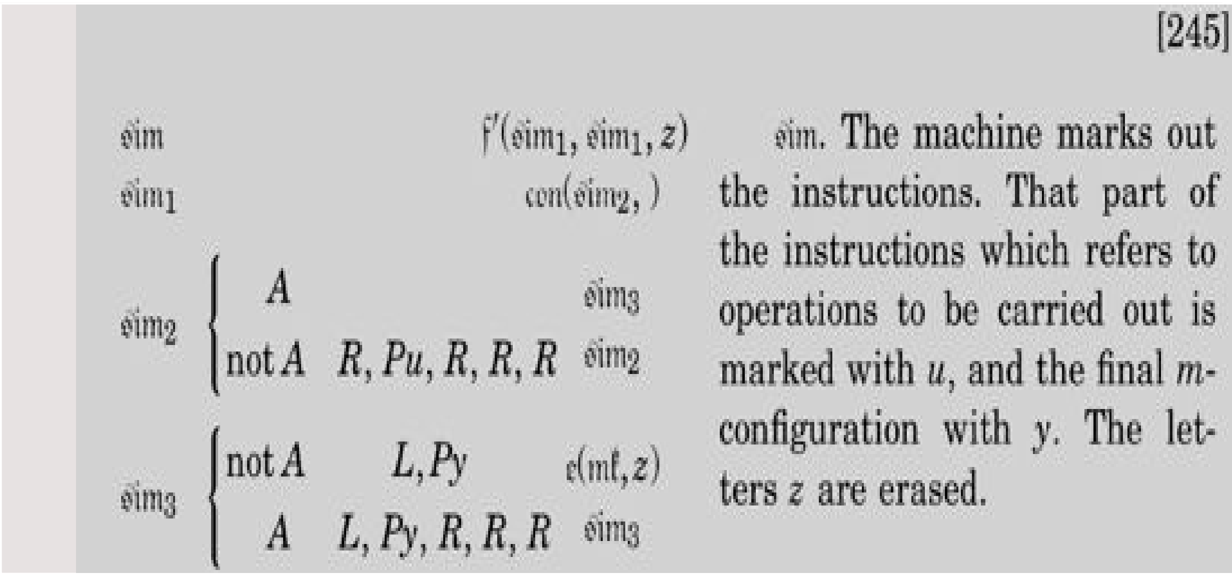
:		D	y	A	y	D	y	•••

□□□□□$\mathfrak{kmp}$□□□□□$\mathfrak{cpe}$□□□□□□□□□□$\mathfrak{sim}$□□□□□□x□y□□□□□□□□□□□z□
□□□□□□z□□$\mathfrak{A}$□□□□□□□□□□□□□□□□□□□□□□□

$\mathfrak{anf}$. Taking the long view, the last instruction relevant to the last configuration is found. It can be recognised afterwards as the instruction following the last semi-colon marked *z.* → $\mathfrak{sim}$

anf□□z□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□→sim□

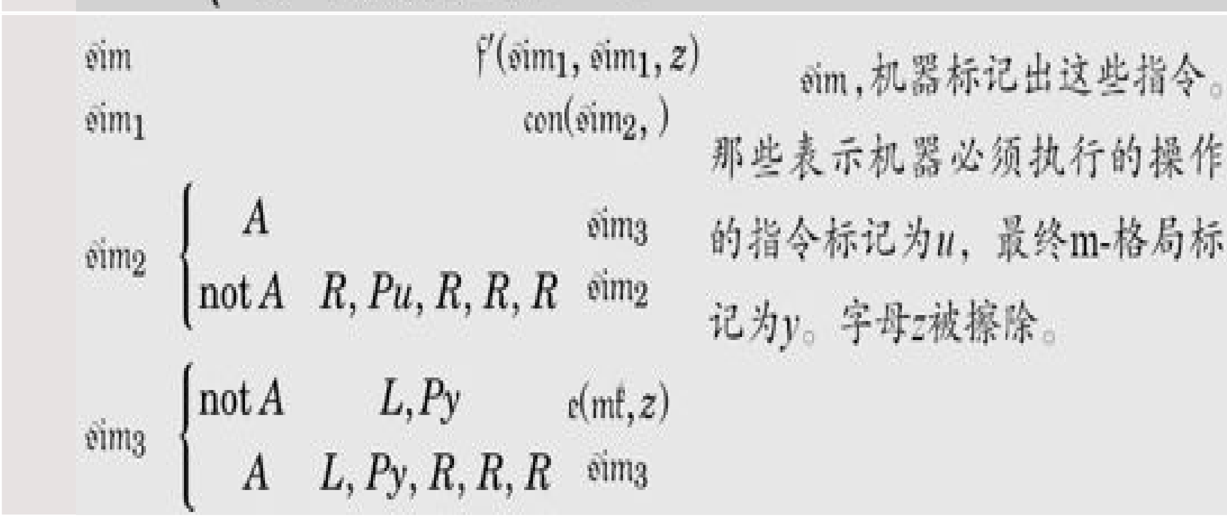
□□□□□□□□□□□□□□□□□□□□□□z□□□□□□□□□□□□□□□□□□□□□□m-□□sim□□□□f′□□m-□□□□□□□□□□□□□□□□L□N□R□□□□□m-□□□



[245]

sim			f′(sim$_1$, sim$_1$, z)
sim$_1$			con(sim$_2$,)
sim$_2$	A		sim$_3$
	not A	R, Pu, R, R, R	sim$_2$
sim$_3$	not A	L, Py	e(mf, z)
	A	L, Py, R, R, R	sim$_3$

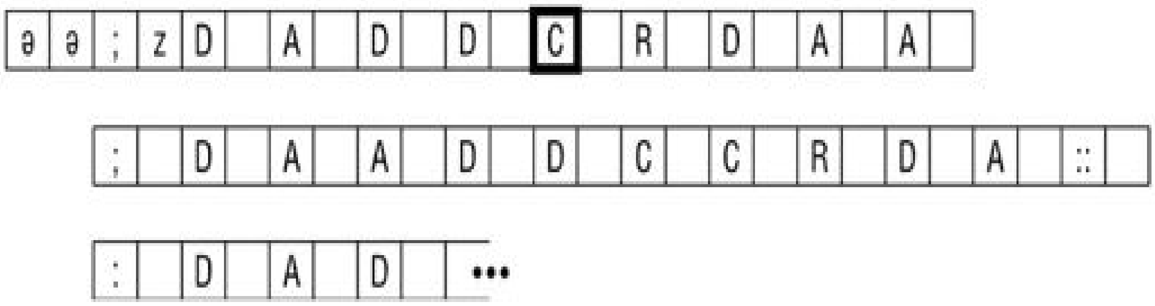
sim. The machine marks out the instructions. That part of the instructions which refers to operations to be carried out is marked with u, and the final m-configuration with y. The letters z are erased.



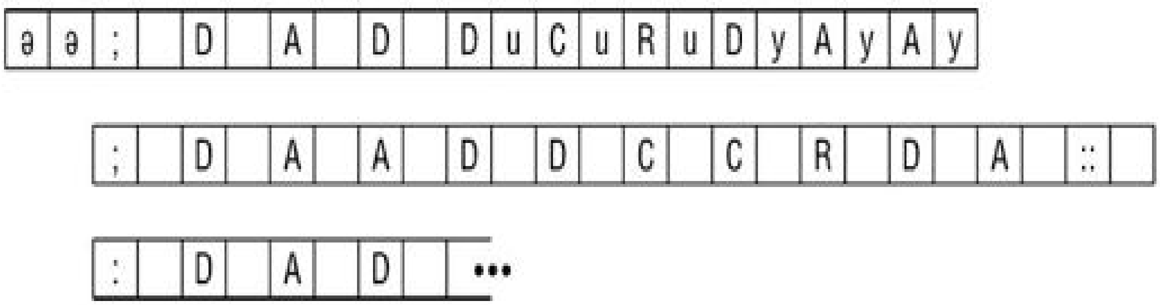
sim			f′(sim$_1$, sim$_1$, z)
sim$_1$			con(sim$_2$,)
sim$_2$	A		sim$_3$
	not A	R, Pu, R, R, R	sim$_2$
sim$_3$	not A	L, Py	e(mf, z)
	A	L, Py, R, R, R	sim$_3$

sim，机器标记出这些指令。那些表示机器必须执行的操作的指令标记为u，最终m-格局标记为y。字母z被擦除。

m-□□sim$_1$□□□□□□□□□□□□□□□con□□□□□□□□m-□□□□□□□□□□□□□□□□□□□□□□□□□□□□



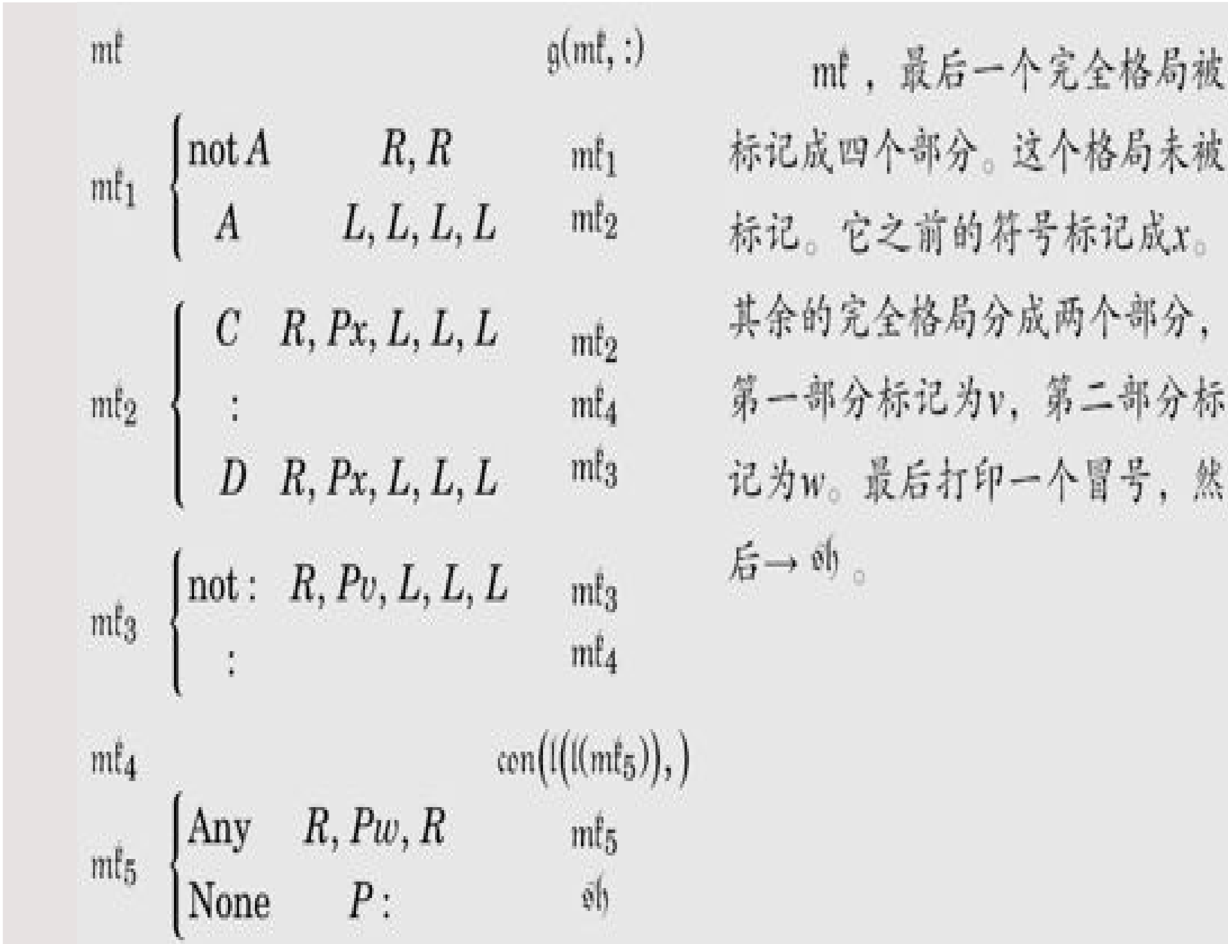
m-□□$\mathfrak{sim}_2$□□□□□□□□□□□·□□□□□□□□□□□□□□u□□□□□□□□□□□m-□□$\mathfrak{sim}_2$□$\mathfrak{sim}_3$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□□□$\mathfrak{e}$□□□$\mathfrak{mk}$□□□□□□□z□



m-□□m$\mathfrak{mk}$□□□□□mf□□□□□mk□□□□□“□□”□□□□□□□□□□□□□□□□□$\mathfrak{g}$□□□□□□□□$\mathfrak{q}$□□□□□□□□□□□$\mathfrak{mk}_1$□□□□$\mathfrak{mk}$□

m-config.	Symbol	Behaviour	Final m-config.
$\mathfrak{mk}$			$\mathfrak{g}(\mathfrak{mk}, :)$
$\mathfrak{mk}_1$	not A	R, R	$\mathfrak{mk}_1$
	A	L, L, L, L	$\mathfrak{mk}_2$
$\mathfrak{mk}_2$	C	R, Px, L, L, L	$\mathfrak{mk}_2$
	$:$		$\mathfrak{mk}_4$
	D	R, Px, L, L, L	$\mathfrak{mk}_3$
$\mathfrak{mk}_3$	not $:$	R, Pv, L, L, L	$\mathfrak{mk}_3$
	$:$		$\mathfrak{mk}_4$
$\mathfrak{mk}_4$			$\mathfrak{con}(\mathfrak{l}(\mathfrak{l}(\mathfrak{mk}_5)),)$
$\mathfrak{mk}_5$	Any	R, Pw, R	$\mathfrak{mk}_5$
	None	$P:$	$\mathfrak{sh}$

$\mathfrak{mk}$. The last complete configuration is marked out into four sections. The configuration is left unmarked. The symbol directly preceding it is marked with x. The remainder of the complete configuration is divided into two parts, of which the first is marked with v and the last with w. A colon is printed after the whole. $\rightarrow \mathfrak{sh}$.



格局	符号	操作	最终格局
mk			g(mk, :)
mk_1	not A	R, R	mk_1
	A	L, L, L, L	mk_2
mk_2	C	R, Px, L, L, L	mk_2
	:		mk_4
	D	R, Px, L, L, L	mk_3
mk_3	not :	R, Pv, L, L, L	mk_3
	:		mk_4
mk_4			con(l(l(mk_5)),)
mk_5	Any	R, Pw, R	mk_5
	None	P:	sh

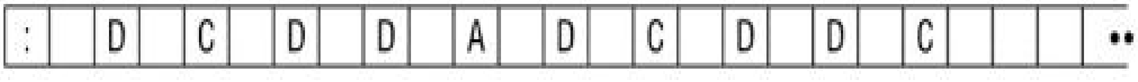
mk，最后一个完全格局被标记成四个部分。这个格局未被标记。它之前的符号标记成x。其余的完全格局分成两个部分，第一部分标记为v，第二部分标记为w。最后打印一个冒号，然后→ sh。

m-□□mk□□g □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□F-□□□□□□□□□□D□□0□□□□□C□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□□□□□□□□D□□□□□A□□□□

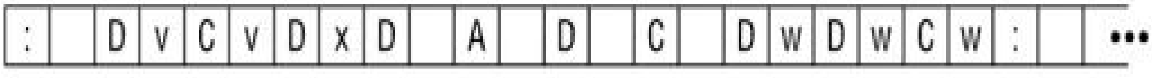
m-□□mk_1□□□□□□□□□□□□□m-□□□□□□□A□□□□□□□□□□□□□m-□□□□□□□□□□□□□□□□□□□□x□□□□mk_3□□□□□□□□□□□□v□

mk_3□□□□□□□□mk_4□□□□□□□□□mk_4□□con□□m-□□□□□□□□□□□□□□C□□□□□□□□□□□□□□□□□□□□□□□w□□□□mk_5□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□



□□□□□□□□□□□0□DC□□□□□□□D□□□□□□□□□□□□□□□□□□□□□□□□□q_1□DA□□□□□□□□□□□0□DC□□□□□□□□□D□□□□0□DC□□mk□□□□□□□□□



□□□□□□□□□□□□□m-□□DA□□□□DC□□□□

□□□□□□□□□□m-□□□□□□□□□□□□□□□□□□□□□□□v□x□w□□□□□□□□□□□□□

ə	ə	;		D		A		D		D	u	C	u	R	u	D	y	A	y	A	y

;		D		A		A		D		D		C		C		R		D		A		::	

:		D		A		D		:		…

□□□□□□□□□□□□□□□□□□□□□m-□□□□□□□*u*□*y*□□□□□□□□□□□□□*v*□*x*□*w*□

□□□□□□□□□□□□□0□1□□□□□□□□□0□1□□□□□□□□□□□□□□0□1□□□□□□□□□□□□0□1□□□□□□□□□□□□□□□□□□□□0□1□□□□𝔰𝔥 □□□□□"□□"□□□□□

m-config.	symbol	operations	final m-config.
$\mathfrak{sh}$			$\mathfrak{f}(\mathfrak{sh}_1, \mathfrak{inst}, u)$
$\mathfrak{sh}_1$		L, L, L	$\mathfrak{sh}_2$
$\mathfrak{sh}_2$	D	R, R, R, R	$\mathfrak{sh}_2$
	not D		$\mathfrak{inst}$
$\mathfrak{sh}_3$	C	R, R	$\mathfrak{sh}_4$
	not C		$\mathfrak{inst}$
$\mathfrak{sh}_4$	C	R, R	$\mathfrak{sh}_5$
	not C		$\mathfrak{pe}_2(\mathfrak{inst}, 0, :)$
$\mathfrak{sh}_5$	C		$\mathfrak{inst}$
	not C		$\mathfrak{pe}_2(\mathfrak{inst}, 1, :)$

$\mathfrak{sh}$. The instructions (marked u) are examined. If it is found that they involve "Print 0" or "Print 1", then 0: or 1: is printed at the end.

$\mathfrak{sh}$		$\mathfrak{f}(\mathfrak{sh}_1, \mathfrak{inst}, u)$	$\mathfrak{sh}$，表示检测标记为u的指令。如果指令中包含“打印0”或“打印1”，那么在末尾打印0:或1:。
$\mathfrak{sh}_1$		L, L, L	$\mathfrak{sh}_2$
$\mathfrak{sh}_2$	D	R, R, R, R	$\mathfrak{sh}_2$
	not D		$\mathfrak{inst}$
$\mathfrak{sh}_3$	C	R, R	$\mathfrak{sh}_4$
	not C		$\mathfrak{inst}$
$\mathfrak{sh}_4$	C	R, R	$\mathfrak{sh}_5$
	not C		$\mathfrak{pe}_2(\mathfrak{inst}, 0, :)$
$\mathfrak{sh}_5$	C		$\mathfrak{inst}$
	not C		$\mathfrak{pe}_2(\mathfrak{inst}, 1, :)$

□□□$\mathfrak{sh}$□□□□□□□□□u□$\mathfrak{sh}_1$□□□D□□□□□D□□□□□□□□□m-□□□□$\mathfrak{inst}$□

□□□□□□□□□□□□□□□$\mathfrak{sh}_2$□□$\mathfrak{sh}_3$□□□□□□□□□$\mathfrak{sh}_2$□□□□□□□$\mathfrak{sh}_3$□$\mathfrak{sh}_4$□$\mathfrak{sh}_5$□□□□□□□□□□□DC□□□□□0□□□DCC□□□□□1□□□□□□□□$\mathfrak{pe}_2$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

ə	ə	;		D		A		D		D	u	C	u	R	u	D	y	A	y	A	y

;		D		A		A		D		D		C		C		R		D		A		::	

:		D		A		D		:		0		:		…

□□□□□$\mathfrak{sh}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□8□m-□□□$\mathfrak{sh}_6$□$\mathfrak{sh}_{13}$□□□□2□9□□

□□□□□0□1□□□□□□□□□□□□□□□□□□□$\mathfrak{inst}$[□□□□instruction□□□□□□□□□□□□instigate□□□□□□□□]□□□□□□□□□□□□□$\mathcal{M}$□□□□□□□□□□□□□□□□□□□□□□□□□x□v□

w[illegible]m-[illegible]y[illegible]m-[illegible]u[illegible]

$\mathfrak{inst}$ [illegible] $\mathfrak{q}$ [illegible] $\mathfrak{g}$ [illegible] $\mathfrak{ec}_5$ [illegible] $\mathfrak{ec}_5$ [illegible]

[246]

$\mathfrak{inst}$			$\mathfrak{g}(\mathfrak{l}(\mathfrak{inst}_1), u)$	$\mathfrak{inst}$. The next complete configuration is written down, carrying out the marked instructions. The letters u, v, w, x, y are erased. → $\mathfrak{anf}$.
$\mathfrak{inst}_1$	α	R, E	$\mathfrak{inst}_1(\alpha)$	
$\mathfrak{inst}_1(L)$			$\mathfrak{ce}_5(\mathfrak{ov}, v, y, x, u, w)$	
$\mathfrak{inst}_1(R)$			$\mathfrak{ce}_5(\mathfrak{ov}, v, x, u, y, w)$	
$\mathfrak{inst}_1(N)$			$\mathfrak{ec}_5(\mathfrak{ov}, v, x, y, u, w)$	
$\mathfrak{ov}$			$\mathfrak{e}(\mathfrak{anf})$	

$\mathfrak{inst}$			$\mathfrak{g}(\mathfrak{l}(\mathfrak{inst}_1), u)$	$\mathfrak{inst}$ ，写入的下一个完全格局，用于执行被标记的指令。字母u、v、w、x、y被擦除。最后→ $\mathfrak{anf}$。
$\mathfrak{inst}_1$	α	R, E	$\mathfrak{inst}_1(\alpha)$	
$\mathfrak{inst}_1(L)$			$\mathfrak{ce}_5(\mathfrak{ov}, v, y, x, u, w)$	
$\mathfrak{inst}_1(R)$			$\mathfrak{ce}_5(\mathfrak{ov}, v, x, u, y, w)$	
$\mathfrak{inst}_1(N)$			$\mathfrak{ec}_5(\mathfrak{ov}, v, x, y, u, w)$	
$\mathfrak{ov}$			$\mathfrak{e}(\mathfrak{anf})$	

[illegible] $\mathfrak{ce}_4$ [illegible] $\mathfrak{ce}_5$ [illegible] $\mathfrak{ce}_3$ [illegible]

$\mathfrak{ce}_4(\mathfrak{B}, \alpha, \beta, \gamma, \delta)$	$\mathfrak{ce}(\mathfrak{ce}_3(\mathfrak{B}, \beta, \gamma, \delta), \alpha)$
$\mathfrak{ce}_5(\mathfrak{B}, \alpha, \beta, \gamma, \delta, \varepsilon)$	$\mathfrak{ce}(\mathfrak{ce}_4(\mathfrak{B}, \beta, \gamma, \delta, \varepsilon), \alpha)$

[illegible] $\mathfrak{ce}_5$ [illegible]α[illegible]β[illegible]

m-[illegible] $\mathfrak{inst}$ [illegible] $\mathfrak{g}$ [illegible] $\mathfrak{g}$ [illegible]u[illegible]L[illegible]R[illegible]N[illegible]m-[illegible] $\mathfrak{inst}_1$ [illegible] $\mathfrak{inst}_1(L)$ [illegible] $\mathfrak{inst}_1(R)$ [illegible] $\mathfrak{inst}_1(N)$ [illegible] $\mathfrak{inst}_1$ [illegible]

$$\mathfrak{inst}_1 \quad \begin{cases} L & R, E & \mathfrak{ce}_5(\mathfrak{ov}, v, y, x, u, w) \\ R & R, E & \mathfrak{ce}_5(\mathfrak{ov}, v, x, u, y, w) \\ N & R, E & \mathfrak{ce}_5(\mathfrak{ov}, v, x, y, u, w) \end{cases}$$

[illegible]*v*[illegible]*w*[illegible]*v*[illegible]*x*[illegible]*w*[illegible]

$\mathfrak{ec}_5$[illegible]

[illegible]m-[illegible]/[illegible]/[illegible]

[illegible]/[illegible]/[illegible]m-[illegible]

[illegible]/[illegible]m-[illegible]/[illegible]

[illegible]m-[illegible]m-[illegible]

$\mathfrak{ec}_5$[illegible]$\mathfrak{ov}$ [[illegible]over[illegible]][illegible] $\mathfrak{e}$ [illegible]E-[illegible]$\mathfrak{anf}$ [illegible]

ə	ə	;		D		A		D		D		C		R		D		A		A	

;		D		A		A		D		D		C		C		R		D		A		::	

:		D		A		D		:		0		:		D		C		D		A		A		…

[illegible]*DC*[illegible]0[illegible]*DAA*[illegible]m-[illegible]q_2[illegible]

[illegible]$\mathfrak{con}$[illegible]0[illegible]1[illegible]0[illegible]1[illegible]

[illegible]“[illegible]” [4]

[illegible]

[illegible]·[illegible]

[1] □□□□□□·□□□□□“Recursive Unsolvability of a Problem of Thue”□*The Journal of Symbolic Logic*□Vol. 12□No. 1□Mar. 1947□□1-11□□□□□□□□□□□□Martin Davis□ed.□*The Undecidable*□Raven Press□1965□□293-303□□□□□B. Jack Copeland□ed.□*The Essential Turing*□Oxford University Press□2004□□97-101□

[2] □□□□·□□□□“Corrections to Turing’s Universal Computing Machine”□in C. Jack Copeland□ed.□*The Essential Turing*□103-124□□□□□□□□□□□□□□□□□□□□□□□□□□□□

[3] □□□□□□□“Recursive Unsolvability of a Problem of Thue”□7□

[4] John P. Burgess□George S. Boolos□John P. Burgess□Richard C. Jeffrey□□□Computability and Logic, fourth edition□Cambridge University Press□2002□□□□□□□□□□□

[illegible]10[illegible]
[illegible]

[illegible]“[illegible]”[illegible]·[illegible]——[illegible]

[illegible]·[illegible]1937[illegible]1956[illegible]

> [illegible][1]

[illegible]1936[illegible]1956[illegible]1950[illegible]

> [illegible][2]

[illegible]“[illegible]”[illegible]

[illegible]·[illegible][3][illegible][4][illegible]

[illegible]·[illegible]·[illegible]1935[illegible]4[illegible]·[illegible][5] 1936[illegible][6][illegible]·[illegible][7][illegible]“[illegible]”[illegible]·[illegible]1937[illegible]6[illegible]1[illegible]

1938[illegible]7[illegible]·[illegible]1500[illegible]·[illegible]·[illegible]·[illegible]·[illegible]·[illegible]1957[illegible]53[illegible]1938[illegible]·[illegible]·[illegible]

[illegible]…[illegible]·[illegible]1939[illegible]“[illegible]”……1939[illegible]·[illegible][8]

[illegible]1944[illegible]9[illegible]·[illegible]ENIAC[illegible]·[illegible]1919—1995[illegible]·[illegible]·[illegible]1907—1980[illegible]30[illegible]EDVAC[illegible]

[illegible]·[illegible]1944[illegible]1945[illegible]EDVAC[illegible][9]

[illegible]1945[illegible]6[illegible]30[illegible]·[illegible]·[illegible]“EDVAC[illegible]”[10][illegible]·[illegible]ENIAC[illegible]“[illegible]·[illegible]”[illegible]

“EDVAC[illegible]”[illegible]“[illegible]”[11][illegible]·[illegible]·[illegible]1894—1964[illegible]1948[illegible]17[illegible]·[illegible]

[illegible]·[illegible]·[illegible]·[illegible]1943[illegible]1944[illegible]

[illegible]·[illegible]·[illegible]“[illegible]”[illegible]·[illegible][12]

20[illegible]40[illegible]·[illegible]1946[illegible]·[illegible]“[illegible]……[illegible]‘[illegible]’[illegible]”[13]

[illegible]·[illegible]

[illegible]Colossus[illegible]1943[illegible]·[illegible]1905—1998[illegible]20[illegible]30[illegible][14][illegible]“[illegible]”[15][illegible][16]

[illegible]NPL[illegible]1944[illegible]NPL[illegible]·[illegible]1887—1962[illegible][17][illegible]“[illegible]”[illegible]

[illegible]J. R. [illegible]1945[illegible]6[illegible]NPL[illegible][18][illegible]“[illegible]”[illegible]“[illegible]”[illegible]ACE[illegible]“[illegible]”[illegible]·[illegible]

[illegible]·[illegible]EDVAC[illegible]1945[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]·[illegible]‘[illegible]EDVAC[illegible]’[illegible]”[illegible][19]

[illegible]1MHz[illegible]1024[illegible]1[illegible]32[illegible]32[illegible]32[illegible]“[illegible]2[illegible]”[illegible][20]

[illegible]RISC[illegible]——[illegible]——[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]BURY[illegible]“[illegible]”[illegible]UNBURY[illegible][21]

[illegible]1947[illegible]2[illegible]20[illegible]“[illegible]ACE[illegible]……[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible][22] [illegible]

> [illegible][23]

[illegible]

> [illegible][24]

[illegible]

[illegible][25]

[illegible]1947[illegible]9[illegible]ACE[illegible]NPL[illegible]ACE[illegible]1950[illegible]

[illegible]1945[illegible]·[illegible]Mark I[illegible]1948[illegible]6[illegible]Mark I[illegible]“[illegible]EDVAC[illegible]”[illegible][26]

[illegible]9[illegible]

[illegible]Mark I[illegible]1951[illegible]“[illegible]”[illegible]“[illegible]”[27]

[illegible]Mark I[illegible]CRT[illegible]1946[illegible]12[illegible]F. C. [illegible]CRT[illegible]0[illegible]1[illegible]CRT[illegible]1947[illegible]CRT[illegible]2048[illegible]

Mark I[illegible]40[illegible]20[illegible]5[illegible]CRT[illegible]5[illegible]32[illegible]Mark I[illegible]32[illegible]

/E@A:SIU $\frac{1}{2}$ DRJNFCKTZLWHYPQOBG"MXV£

[illegible]“[illegible]”[illegible]

[illegible]“[illegible]”[illegible]230[illegible]57[illegible]5[illegible]“[illegible]”[illegible]241[illegible]125[illegible]

[illegible]π[illegible]e[illegible]·[illegible]

[illegible]……[illegible]“[illegible]”[illegible]

[illegible]

[illegible]

[illegible]C[illegible]rand[illegible]0[illegible]32767[illegible]rand[illegible]1[illegible]rand[illegible]rand[illegible]C[illegible][28][illegible]

```
int seed = 1;

int rand ()
{
   return ((seed = seed * 214013 + 2531011) >> 16) & 32767;
}
```

[illegible]rand[illegible]214 013[illegible]2 531 011[illegible]rand[illegible]32[illegible]32[illegible]1[illegible][29][illegible]16[illegible]65 536[illegible]32 767[illegible]AND[illegible]15[illegible]0[illegible]32 767[illegible]

[illegible]rand[illegible]1[illegible]

41

18 467

6 334

26 500

19 169

…

[illegible]rand[illegible]41[illegible]30 546[illegible]rand[illegible]41[illegible]

[illegible]rand[illegible]

[illegible]

[illegible]·[illegible]·[illegible]“[illegible]”[30][illegible]RNG[illegible]

[illegible]·[illegible]Mark I[illegible]

[illegible][31]

[illegible]RNG[illegible]0[illegible]1[illegible]

[illegible]RNG[illegible]RNG[illegible]

[illegible]π[illegible]π[illegible]

[illegible]

8. *Application of the diagonal process.*

It may be thought that arguments which prove that the real numbers are not enumerable would also prove that the computable numbers and sequences cannot be enumerable*. It might, for instance, be thought that the limit of a sequence of computable numbers must be computable.

* *Cf.* Hobson, *Theory of functions of a real variable* (2nd ed., 1921), 87, 88.

8. [illegible]

[illegible]*[illegible]

* *Cf*. Hobson[illegible] *Theory of functions of a real variable*[illegible]2nd ed.[illegible]1921[illegible]87[illegible]88[illegible]

[illegible]·[illegible]1874[illegible]20[illegible][32][illegible]E. W. [illegible]

[illegible]

[illegible]

This is clearly only true if the sequence of computable numbers is defined by some rule.

[illegible]

[illegible]

Or we might apply the diagonal process.

[illegible]

[illegible] 23 [illegible]

"If the computable sequences are enumerable let α_n be the n-th computable sequence, and let $\phi_n(m)$ be the m-th figure in α_n.

"[illegible] α_n [illegible] n [illegible] $\phi_n(m)$ [illegible] α_n [illegible] m [illegible]

[illegible] 0 [illegible] 1 [illegible] ϕ [illegible]

$$\alpha_1 = \phi_1(1)\ \ \phi_1(2)\ \ \phi_1(3)\ \ \phi_1(4)\ \ \ldots$$
$$\alpha_2 = \phi_2(1)\ \ \phi_2(2)\ \ \phi_2(3)\ \ \phi_2(4)\ \ \ldots$$
$$\alpha_3 = \phi_3(1)\ \ \phi_3(2)\ \ \phi_3(3)\ \ \phi_3(4)\ \ \ldots$$
$$\ldots$$

Let β be the sequence with $1 - \phi_n(n)$ as its n-th figure.

[illegible] β [illegible] $1-\phi_n(n)$ [illegible] n [illegible]

[illegible] β [illegible] 0 [illegible] 1 [illegible]

$$\beta = (1-\phi_1(1))\ (1-\phi_2(2))\ (1-\phi_3(3))\ (1-\phi_4(4))\ldots$$

Since β is computable, there exists a number K such that $1 - \phi_n(n) = \phi_K(n)$ all n.

[illegible] β [illegible] K [illegible] $1 - \phi_n(n) = \phi_K(n)$ [illegible] n [illegible]

[illegible] K [illegible] α_K [illegible]

$$\beta = \alpha_K = \phi_K(1)\ \ \phi_K(2)\ \ \phi_K(3)\ \ \phi_K(4)\ \ldots$$

[illegible] n, [illegible] $1-\phi_n(n) = \phi_K(n)$ 或 $1 = \phi_K(n) + \phi_n(n)$ [illegible]

[illegible] β [illegible]

Putting $n=K$,

[illegible] $n=K$,

[illegible] $1 = \phi_K(K) + \phi_K(K)$ [illegible]

we have $1 = 2\phi_K(K)$, *i.e.* 1 is even. This is impossible. The computable sequences are therefore not enumerable."

[illegible]1=2$\phi_K(K)$[illegible]1[illegible]”

[illegible]β[illegible]

[illegible]

The fallacy in this argument lies in the assumption that β is computable.

[illegible]β[illegible]

[illegible]β[illegible]β[illegible]

[illegible]

[illegible]

[illegible]

1

2

3

4

5

…

[illegible]

[illegible]

[illegible]2[illegible]0[illegible]1[illegible]“[illegible]”[illegible]“[illegible]”[illegible]1[illegible]1/2[illegible]1[illegible]0[illegible]

.10000000…

[illegible]0[illegible]0[illegible]1[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]0[illegible]1[illegible]m-[illegible]

[illegible]

[illegible]m-[illegible]m-[illegible]m-[illegible]0[illegible]1[illegible]31 334 317[illegible]313 324 317[illegible]313 325 317[illegible]

[illegible]

1

2

3

4

5

…

313 325 317 ←[illegible]0[illegible]

…

3 133 225 317←[illegible]1[illegible]

…

[illegible]

[illegible]β[illegible]β[illegible]β[illegible]

> It would be true if we could enumerate the computable sequnces by finite means, but the problem of enumerating computable sequences is equivalent to the problem of finding out whether a given numer is the D.N of a circle-free machine, and we have no general process for doing this in a finite number of steps.

> [illegible]

[illegible]

> In fact, by applying the diagnoal process argument correctly, we can show that there cannot be any such general process.

> [illegible]

[illegible]β[illegible]

[illegible]

> The simplest and most direct proof of this is by showing that, if this general process exists, then there is a machine which computes β.

> [illegible]β[illegible]

[illegible] β [illegible] β [illegible]

This proof, although perfectly sound, has the disadvantage that it may leave the reader with a feeling that "there must be something wrong".

[illegible]

[illegible]

The proof which I shall give has not this disadvantage, and gives a certain insight into the significance of the idea "circle-free".

[illegible]

[illegible]

[illegible] β [illegible]

It depends not on constructing β, but on constructing β', whose n-th figure is $\phi_n(n)$.

[illegible] β [illegible] β' [illegible] β' [illegible] n [illegible] $\phi_n(n)$ [illegible]

[illegible] 57 [illegible] β [illegible] β' [illegible] β' [illegible] 1 [illegible] n [illegible] n [illegible] 1 [illegible] β' [illegible] n [illegible]

[illegible] β' [illegible]

[illegible] halting problem [illegible] 1958 [illegible] [33] [illegible]

[illegible]

[247]

Let us suppose that there is such a process; that is to say, that we can invent a machine $\mathcal{D}$ which, when supplied with the S.D of any computing machine $\mathcal{M}$ will test this S.D and if $\mathcal{M}$ is circular will mark the S.D with the symbol *"u"* and if it is circle-free will mark it with *"s"*

By combining the machines $\mathcal{D}$ and $\mathcal{U}$ we could construct a machine $\mathcal{H}$ to compute the sequence β'.

The machine $\mathcal{D}$ may require a tape. We may suppose that it uses the *E*-squares beyond all symbols on *F*-squares, and that when it has reached its verdict all the rough work done by $\mathcal{D}$ is erased.

The machine $\mathcal{H}$ has its motion divided into sections. In the first N - 1 sections, among other things, the integers 1, 2,…, N - 1 have been written down and tested by the machine $\mathcal{D}$.

A certain number, say *R*(*N* - 1), of them have been found to be the D.N's of circle-free machines.

□□□□□□*R*(*N*-1)□□□□□□□□□□□□□

R□□□□□□□□□□□□□□□□□□□□□*R*□□□□□□□□□□□□□□□□□□□□□□□□□□

In the *N*-the section the machine $\mathcal{D}$ tests the number *N.* If *N* is satisfactory, *i.e.*, if it is the D.N of a circle-free machine, then *R*(*N*) = 1 + R(*N* - 1) and the first *R*(*N*) figures of the sequence of which a D.N is *N* are calculated.

□□*N*□□□□□□□$\mathcal{D}$□□□*N*□□□□*N*□□□□□□□□□□□□□□□□□□□□□□□□ *R*(*N*)=1+*R*(*N*-1)□□□□□□□□□□□□*N*□□□□□*R*(*N*)□□□□

□□□□□□□□*N*□3 133 225 317□□*R*(*N*)□1□□$\mathcal{D}$□□□*N*□□□□□□□□□□□□□□□□□*R*(*N*)□□ 2□□□$\mathcal{U}$□□□3 133 225 317□□□□□□□□□□□□□□□□□□1□$\mathcal{H}$□□□□□□□□□□□□□*β*′□ □□□□□□□□□□□□□□

The *R*(*N*)-th figure of this sequence is written down as one of the figures of the sequence *beta;*′ computed by $\mathcal{H}$

□□*H*□□□□□□□□□□□□*R*(*N*)□□□□□□□□□□□*β*′□□□□

□□□□□□□□□□□□*N*□□□□□□□□□□□□□□□□□□

If *N* is not satisfactory, then *R*(*N*) = *R*(*N* - 1) and the machine goes on to the (*N* + 1)-th section of its motion.

□□*N*□□□□□□□□□□*R*(*N*)=*R*(*N*-1)□□□□□□□*N*+1□□□□□□□□□

□□□□□□□□□$\mathcal{H}$□□□$\mathcal{H}$□□□□□□□□□ *R*(*N*)□□

□□□□□□□□□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□□□$\mathcal{D}$□□□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□□□□□□□ □$\mathcal{D}$□

From the construction of $\mathcal{H}$ we can see that $\mathcal{H}$ is circle-free. Each section of the motion of $\mathcal{H}$ comes to an end after a finite number of steps. For, by our assumption about $\mathcal{D}$, the decision as to whether *N* is satisfactory is reached in a finite number of steps. If *N* is not satisfactory, then the *N*-th section is finished. If *N* is satisfactory, this means that the machine $\mathcal{M}$(*N*) whose D.N is *N* is circle-free, and therefore its *R*(*N*)-th figure can be calculated in a finite number of steps. When this figure has been calculated and written down as the *R*(*N*)-th figure of *β*′, the *N*-th section is finished. Hence $\mathcal{H}$ is circle-free.

□$\mathcal{H}$□□□□□□□□□$\mathcal{H}$□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathcal{D}$□□□□□□□□□□□□□N□□□□□□□□□□□N□□□□□□□□□□□N□□□□□□□□N□□□□□□□□□□□□□□□□N□□□□$\mathcal{M}$□□□□□□□□□□□$R(N)$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□β'□□$R(N)$□□□□□□□□N□□□□□□□□□$\mathcal{H}$□□□□□□□

$\mathcal{H}$□□□□□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□□□K□□□□□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□□□□□□□

Now let K be the D.N of $\mathcal{H}$. What does $\mathcal{H}$ do in the K-th section of its motion? It must test whether K is satisfactory, giving a verdict *"s"* or *"u"*. Since K is the D.N of $\mathcal{H}$ and since $\mathcal{H}$ is circle-free, the verdict cannot be *"u"*

□K□$\mathcal{H}$□□□□□□$\mathcal{H}$□□K□□□□□□□□□□□□□□□□□□□□□□□K□□□□□□□□□□□□□□□□□□□s□u□□□K□$\mathcal{H}$□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□□□□□□□□□u□

□□□□□

On the other hand the verdict cannot be *"s"*.

□□□□□□□□□□□□□□□□□□s□

□□□□□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□$\mathcal{H}$□□□□□□□K□□□□□□□□□□□□□□□□□□$\mathcal{H}$□□□□□□1□K-1□□□□□□□□□□□□□□□□□□□□□□□□□□$R(K\text{-}1)$□□□β'□□$R(K\text{-}1)$□□□□□□□□□□□□

β'□□$R(K)$□□□□□□□□□□□□□□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□K□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□□□

For if it were, then in the K-th section of its motion $\mathcal{H}$ would be bound to compute the first $R(K - 1) + 1 = R(K)$ figures of the sequence computed by the machine with K as its D.N and to write down the $R(K)$-th as a figure of the sequence computed by $\mathcal{H}$. The computation of the first $R(K) - 1$ figures would be carried out all right, but the instructions for calculating the $R(K)$-th would amount to "calculate the first $R(K)$ figures computed by H and write down the $R(K)$-th". This $R(K)$-th figure would never be found.

□□□□□□s□□□□□□$\mathcal{H}$□□K□□□□□□□□□□□□□□□K□□□□□□□□□□□□□□□□□□□□$R(K\text{-}1)+1=R(K)$□□□□□□□□$R(K)$□□□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□$R(K\text{-}1)$□□□□□□□□□□□□□□□□□□□□□□□□□□$R(K)$□□□□□□□□□□□□□"□□□H□□□□$R(K)$□□□□□□□□□□□$R(K)$□□□"□□□□□$R(K)$□□□□□□□□□□□□□□

□□□□□□□□□□H□□□□$\mathcal{H}$□□

□□□□□□□□□□□□□□□□□□$\mathcal{H}$□□□1/3□π□2□□□□□□□□□□□□□□□□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□□□□□□□□K□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathcal{H}$□□□□□□□□□□□□□□□□□□

[illegible]

I.e., $\mathcal{H}$ is circular, contrary both to what we have found in the last paragraph and to the verdict *"s".* Thus both verdicts are impossible and we conclude that there can be no machine $\mathcal{D}$.

[illegible]

[illegible]

[illegible]

[illegible]

[248]

We can show further that *there can be no machine* $\mathcal{E}$ *which, when supplied with the S.D of an aribtrary machine* $\mathcal{M}$, *will determine whether* $\mathcal{M}$ *ever prints a given symbol* (0 *say*).

[illegible]

[illegible]

We will first show that, if there is a machine $\mathcal{E}$, then there is a general process for determining whether a given machine $\mathcal{M}$ prints 0 infinitely often.

[illegible]

[illegible]

Let $\mathcal{M}_1$ be a machine which prints the same sequence as $\mathcal{M}$, except that in the position where the first 0 printed by $\mathcal{M}$ stands, $\mathcal{M}_1$ prints $\bar{0}$. $\mathcal{M}_2$ is to have the first two symbols 0 replaced by $\bar{0}$, and so on. Thus, if $\mathcal{M}$ were to print

$$ABA01AAB0010AB\ldots,$$

then $\mathcal{M}_1$ would print

$$ABA\bar{0}1AAB0010AB\ldots$$

and $\mathcal{M}_2$ would print

$$ABA\bar{0}1AAB\bar{0}0010AB\ldots.$$

[illegible]$\mathcal{M}_1$[illegible]$\mathcal{M}$[illegible]$\mathcal{M}$[illegible]0[illegible]$\mathcal{M}_1$[illegible]$\bar{0}$[illegible]$\mathcal{M}_2$[illegible]0[illegible]$\bar{0}$[illegible]$\mathcal{M}$[illegible]

$$ABA01AAB0010AB\ldots$$[illegible]

[illegible]$\mathcal{M}_1$[illegible]

$$ABA\bar{0}1AAB0010AB\ldots,$$

$\mathcal{M}_2$[illegible]

$$ABA\bar{0}1AAB\bar{0}0010AB\ldots,$$

[illegible]$\mathcal{M}$[illegible]$\mathcal{M}$[illegible]$\mathcal{M}_1$[illegible]$\mathcal{M}_2$[illegible]$\mathcal{F}$[illegible]

Now let $\mathcal{F}$ be a machine which, when supplied with the S.D of $\mathcal{M}$, will write down successively the S.D of $\mathcal{M}$, of $\mathcal{M}_1$, of $\mathcal{M}_2$ … (there is such a machine).

[illegible]$\mathcal{F}$[illegible]$\mathcal{M}$[illegible]$\mathcal{M}$[illegible]$\mathcal{M}_1$[illegible]$\mathcal{M}_2$[illegible]……[illegible]

[illegible]$\mathcal{F}$[illegible]0[illegible]1[illegible]1/3[illegible]

q_1	None	$P0,\ R$	q_2
q_2	None	$P1,\ R$	q_1

[illegible]$\mathcal{M}$[illegible]$\mathcal{M}_1$[illegible]

q_1	None	$P\bar{0}, R$	q_4
q_2	None	$P1, R$	q_1
q_3	None	$P0, R$	q_4
q_4	None	$P1, R$	q_3

□□□□□□□□□□$\mathcal{M}$□$\mathcal{M}_1$□□□□□□□□□□□□□□□□□□m-□□□□□□□□□□□□□□□□□□□0□□□□□□□$\bar{0}$□□□□□□□□□□□□□□□□□□□$\mathcal{M}_2$□□□□

q_1	None	$P\bar{0}, R$	q_4
q_2	None	$P1, R$	q_1
q_3	None	$P\bar{0}, R$	q_6
q_4	None	$P1, R$	q_3
q_5	None	$P0, R$	q_6
q_6	None	$P1, R$	q_5

□□□□□□□□□□□□□□□□□□□□□□□□□□□□q_2□□□□□□□□□□□□□□□□□□□

□□□$\mathcal{F}$□□□□□□□□□□□□□□□$\mathcal{M}$□□□□□□□□□□□□$\mathcal{M}$□□□□□0□□□$\mathcal{M}_1$□$\mathcal{M}_2$□□□□□□□$\mathcal{M}$□□□□□0□□□$\mathcal{M}_1$□$\mathcal{M}_2$□□□□□□□0□□□$\mathcal{M}$□□□0□□□□□$\mathcal{M}_1$□□0□□□□$\mathcal{M}_2$□□□□□□□0□□□$\mathcal{M}$□□□□□□□0□□□$\mathcal{M}_1$□$\mathcal{M}_2$□□□□□□

□□□□□□□□□□$\mathcal{E}$□□□□□□□□□□□□□□□□□□□□□□0□

We combine $\mathcal{F}$ with $\mathcal{E}$ and obtain a new machine, $\mathcal{G}$. In the motion of $\mathcal{G}$ first $\mathcal{F}$ is used to write down the S.D of $\mathcal{M}$, and then $\mathcal{E}$ tests its, :0: is written if it is found that $\mathcal{M}$ never prints 0; then $\mathcal{F}$ writes the S.D of $\mathcal{M}_1$, and this is tested, :0: being printed if and only if $\mathcal{M}_1$ never prints 0, and so on.

□□□□□$\mathcal{E}$□□□$\mathcal{F}$□□□□□□□□□□□$\mathcal{G}$□$\mathcal{G}$□□□□□□□□□□□□$\mathcal{F}$□□$\mathcal{M}$□□□□□□□□□□□$\mathcal{E}$□□□□□□□□$\mathcal{M}$□□□□0□□□□□:0:□□□$\mathcal{F}$□□$\mathcal{M}_1$□□□□□□□□□□□□□□□□$\mathcal{M}_1$□□□□0□□□□:0:□□□□□□□□□□□

$\mathscr{G}$ [illegible] $\mathscr{F}$ [illegible] $\mathscr{M}$ [illegible] $\mathscr{M}_1$ [illegible] $\mathscr{M}_2$ [illegible] $\mathscr{E}$ [illegible] 0 [illegible] 0 [illegible] $\mathscr{G}$ [illegible] 0 [illegible]

[illegible] $\mathscr{M}$ [illegible] 0 [illegible] 0 [illegible] $\mathscr{G}$ [illegible] 0 [illegible] $\mathscr{M}$ [illegible] 0 [illegible] $\mathscr{G}$ [illegible] 0 [illegible]

Now let us test $\mathscr{G}$ with $\mathscr{E}$. If it is found that $\mathscr{G}$ never prints 0, then $\mathscr{M}$ prints 0 infinitely often; if $\mathscr{G}$ prints 0 sometimes, then $\mathscr{M}$ does not print 0 infinitely often.

[illegible] $\mathscr{E}$ [illegible] $\mathscr{G}$ [illegible] $\mathscr{G}$ [illegible] 0 [illegible] $\mathscr{M}$ [illegible] 0 [illegible] $\mathscr{G}$ [illegible] 0 [illegible] $\mathscr{M}$ [illegible] 0 [illegible]

[illegible] $\mathscr{G}$ [illegible] $\mathscr{M}$ [illegible] 0 [illegible] 0 [illegible]

Similarly there is a general process for determining whether $\mathscr{M}$ prints 1 infinitely often. By a combination of these processes we have a process for determining whether $\mathscr{M}$ prints an infinity of figures, *i.e.* we have a process for determining whether $\mathscr{M}$ is circle-free. There can therefore be no machine $\mathscr{E}$.

[illegible] $\mathscr{M}$ [illegible] 1 [illegible] $\mathscr{M}$ [illegible] $\mathscr{M}$ [illegible] $\mathscr{E}$ [illegible]

[illegible] $\mathscr{E}$ [illegible] $\mathscr{D}$ [illegible]——[illegible] $\mathscr{D}$ [illegible]

[illegible]

The expression “there is a general process for determining …” has been used throughout this section as equivalent to “there is a machine which will determine …”. This usage can be justified if and only if we can justify our definition of “computable”.

[illegible]“[illegible]……”[illegible]“[illegible]……”[illegible]“[illegible]”[illegible]

[illegible]

[illegible]“[illegible]”[illegible]

001101010001010001010001000001…

[illegible]“[illegible]”[illegible] IsPrime(n) [illegible] n [illegible] n [illegible] 0 [illegible] n [illegible] IsPrime(n) [illegible] 1 [illegible] IsPrime(n) [illegible] 0 [illegible]

□□□□□□□□□□2□3□5□7□11□13□17□19□23□29□□□

> For each of these “general process” problems can be expressed as a problem concerning a general process for determining whether a given integer *n* has a property *G*(*n*) [*e.g. G*(*n*) might mean “*n* is satisfactory” or “*n* is the Gödel representation of a provable formula”], and this is equivalent to computing a number whose *n*-th figure is 1 if *G* (*n*) is true and 0 if it is false.

> □□□□□□“□□□□”□□□□□□□□□□□□□*n*□□□□□□*G*(*n*)□□*G*(*n*)□□“*n*□□□□□”□□“*n*□□□□□□□□□□□”□□□□□□□□□□□□□□□□□□□*G*(*n*)□□□□□□□□□*n*□□□□1□□□□0□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□0□1□□□□□□□□□□□□□□□□□□□□□□□□□□□·□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□1□0□□

IsPrime(*n*)

IsEven(*n*)

IsOdd(*n*)

IsLessThanTen(*n*)

IsMultipleOfTwentyTwo(*n*)

□□□□□□□□□□□□□□□□□□□*n*=0, 1, 2, 3…□□0□1□□□□□□□□□□□□□□□IsOdd□□□□□□□□□□□□□□□□□□□□□□

□□□$\aleph_0$□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□1□□□□IsPrime□□□□□□□□□□□□1□

{2, 3, 5, 7,11,13,…}

□□□□□□□□□□□□□□□□□□□□□□□□□□□□2□□□□□□□□□□□□□□□□□□□□□□$\aleph_0$□□□□□□□□□□$2^{\aleph_0}$□

□□□□□□□□□□□□□□□$2^{\aleph_0}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\aleph_0$□□□□□□□□□□□□□□□□□□□□□□

□1□ Paul Ceruzzi□*Reckoners: The Prehistory of the Digital Computer, from Relays to the Stored Program Concept, 1935-1945*□Greenwood Press□1983□□43□ Ceruzzi□□□□□Howard Aiken□"The Future of Automatic Computing Machinery"□*Elektronische Rechenanlage und Informationsverarbeitung*□Darmstadt□1956□□33□

□2□ □□·□□□"Computing Machinery and Intelligence"□*Mind*□Vol. LIX□No. 236□October 1950□□441- 2□

□3□ Martin Campbell-Kelly and William Aspray□*Computer: A History of the Information Machine*□Basic Books□1996□□

□4□ □□·□□□□*The Universal Computer: The Road from Leibniz to Turing*□Norton□2000□□

□5□ □□□·□□□□*Alan Turing: The Enigma*□Simon & Schuster□1983□□p. 95□

□6□ □□□□*Alan Turing*□118□

□7□ □□□□*Alan Turing*□124□

□8□ □□□□*Alan Turing*□145□

□9□ Nancy Stern□"John von Neumann's Influence on Electronic Digital Computing, 1944-1946"□*Annals of the History of Computing*□Vol. 2□No. 4□October 1980□□353□

□10□ □□□Brian Randell□ed.□*The Origins of Digital Computers*□Springer□1973□□

□11□ W. S. MacCulloch and W. Pitts□"A Logical Calculus of the Ideas Immanent in Nervous Activity"□*Bulletin of Mathematical Biophysics*□Vol. 5□1943□□115-133□

□12□ □□□□□B. Jack Copeland□ed.□*The Essential Turing: The Ideas that Gave Birth to the Computer Age*□Oxford University Press□2004□□22□□□□□6□"Turing, von Neumann, and the Computer"□□□□□□□Copeland□"Computable Numbers"□□□□□□

□13□ B. Jack Copeland and Diane Proudfoot□"Turing and the Computer" in B. Jack Copeland□ed.□*Alan Turing's Automatic Computing Engine*: *The Master Codebreaker's Struggle to Build the Modern Computer*□Oxford University Press□2005□□116

□14□ □□□□□*Code: The Hidden Language of Computer Hardware and Software*□Microsoft Press□1999□□244

□15□ □□□□*Alan Turing*□268□

□16□ □□□□*Alan Turing*□554□□□5.7□□

[17] □□□□·□□□·□□□□□□□□□□·□□□·□□□□□□□□□□□□□□□□□□□——□□□

[18] B. E. Carpenter and R. W. Doran□*A.M. Turing’s ACE Report of 1946 and Other Papers*□MIT Press□1986□□□□5-6□

[19] *ACE Report*□21□

[20] *ACE Report*□116□

[21] *ACE Report*□76□

[22] *ACE Report*□112-113□

[23] *ACE Report*□112□

[24] *ACE Report*□113-114□

[25] *ACE Report*□123□

[26] Martin Campbell-Kelly□“Programming the Mark I: Early Programming Activity at the University of Manchester”□*Annals of the History of Computing*□Vol. 2□No. 2□April 1980□□134□

[27] Campbell-Kelly□“Programming the Mark I”□147□□□□□□□□□□□□www. alanturing.net/turing_archive/archive/index/manchesterindex.html□

[28] rand.c © 1985-1997□Microsoft Corporation□□□□□□rand □□□□□□□□□□

[29] □□□□□□□*Code*□□□□□□□□□□□□□□□□□□□□□153-154□

[30] □□·□·□□□□*Collected Works, Volume V, Design of Computer, Theory of Automata, and Numerical Analysis* □Macmillan□1963□□768□□□□□□□□□1949□□Monte Carlo□□□□□□

[31] Martin Campbell-Kelly□“Programming the Mark I”□136□

[32] □□□□□□□□□E. W. □□□□1856—1933□□□□□□□□□□□□□□ *The Theory of Functions of a Real Variable and the Theory of Fourier’s Series*□□□□□□□□□□□□1907□□□□□□□□□□□□□□□ 1921□□□□□□□□□□□□□□2□1926□□□□□2□□□□□□□1□1927□□□□□□□□□□1□□□□□□2□□□□□□Harren Press□1950□□Dover Books□1957□□□□□□□□□□□□□□□□□□□□□□□□□□□1□□□□□84□□□85□□□□□□□□□□□□□□

[33] □□·□□□□*Computability and Unsolvability*□McGraw-Hill□1958□□70□□□□□□□□1952□□□□□□□□□□□□□□□Copeland□*The Essential Turing*□40□□□61□□□□□□□□□Stephen Cole Kleene□*Introduction to Metamathematics*□Van Nostrand□1952□□□13□□

[illegible]11[illegible]
[illegible]

[illegible]·[illegible]1[illegible]59[illegible]“[illegible]§9[illegible]”[illegible]9[illegible]·[illegible]“[illegible]”[illegible][1]

[illegible]-[illegible]“[illegible]”[illegible]

[illegible]9[illegible]9[illegible]·[illegible]

> [illegible]“[illegible]”[illegible][2]

[249]

9. The extent of the computable numbers.

No attempt has yet been made to show that the “computable” numbers include all numbers which would naturally be regarded as computable. All arguments which can be given are bound to be, fundamentally, appeals to intuition, and for this reason rather unsatisfactory mathematically. The real question at issue is “What are the possible processes which can be carried out in computing a number?”

The arguments which I shall use are of three kinds.

(*a*) A direct appeal to intuition.

(*b*) A proof of the equivalence of two definitions (in case the new definition has a greater intuitive appeal).

(*c*) Giving examples of large classes of numbers which are computable.

Once it is granted that computable numbers are all "computable", several other propositions of the same character follow. In particular, it follows that, if there is a general process for determining whether a formula of the Hilbert function calculus is provable, then the determination can be carried out by a machine.

I. [Type (*a*)]. This argument is only an elaboration of the ideas of § 1.

Computing is normally done by writing certain symbols on paper. We may suppose this paper is divided into squares like a child's arithmetic book. In elementary arithmetic the two-dimensional character of the paper is sometimes used. But such a use is always avoidable, and I think that it will be agreed that the two-dimensional character of paper is no essential of computation. I assume then that the computation is carried out on one-dimensional paper, *i.e.* on a tape divided into squares. I shall also suppose that the number of symbols which may be printed is finite. If we were to allow an infinity of symbols, then there would be symbols differing to an

arbitrarily small extent†. The effect of this restriction of the number of symbols is not very serious. It is always possible to use sequences of symbols in the place of single symbols. Thus an Arabic numeral such as

[250]

17 or 999999999999999 is normally treated as a single symbol. Similarly in any European language words are treated as single symbols (Chinese, however, attempts to have an enumerable infinity of symbols). The differences from our point of view between the single and compound symbols is that the compound symbols, if they are too lengthy, cannot be observed at one glance. This is in accordance with experience. We cannot tell at a glance whether 9999999999999999 and 999999999999999 are the same.

---

† If we regard a symbol as literally printed on a square we may suppose that the square is $0 \le x \le 1$, $0 \le y \le 1$. The symbol is defined as a set of points in this square, viz. the set occupied by printer's ink. If these sets are restricted to be measurable, we can define the "distance" between two symbols as the cost of transforming one symbol into the other if the cost of moving unit area of printer's ink unit distance is unity, and there is an infinite supply of ink at $x = 2$, $y = 0$. With this topology the symbols form a conditionally compact space.

I. [[illegible](a)][illegible]§1[illegible]

[illegible]†[illegible]17[illegible]999999999999999[illegible]9999999999999999[illegible]999999999999999[illegible]

† [illegible]$0 \le x \le 1$[illegible]$0 \le y \le 1$[illegible]
[illegible]
[illegible]“[illegible]”[illegible]
[illegible]$x = 2$[illegible]$y = 0$[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

The behaviour of the computer at any moment is determined by the symbols which he is observing, and his “state of mind” at that moment. We may suppose that there is a bound *B* to the number of symbols or squares which the computer can observe at one moment. If he wishes to observe more, he must use successive observations. We will also suppose that the number of states of mind which need be taken into account is finite. The reasons for this are of the same character as those which restrict the number of symbols. If we admitted an infinity of states of mind, some of them will be “arbitrarily close” and will be confused. Again, the restriction is not one which seriously affects computation, since the use of more complicated states of mind can be avoided by writing more symbols on the tape.

[illegible]“[illegible]”[illegible]
[illegible]*B*[illegible]
[illegible]
[illegible]“[illegible]”[illegible]
[illegible]

1972[illegible]·[illegible]“[illegible]”[illegible]
[3][illegible]“[illegible]”[illegible]
[illegible]“[illegible]”[illegible]

Let us imagine the operations performed by the computer to be split up into “simple operations” which are so elementary that it is not easy to imagine them further divided. Every such operation consists of some change of the physical system consisting of the computer and his tape. We know the state of the system if we know the sequence of symbols on the tape, which of these are

observed by the computer (possibly with a special order), and the state of mind of the computer. We may suppose that in a simple operation not more than one symbol is altered. Any other changes can be split up into simple changes of this kind. The situation in regard to the squares whose symbols may be altered in this way is the same as in regard to the observed squares. We may, therefore, without loss of generality, assume that the squares whose symbols are changed are always "observed" squares.

Besides these changes of symbols, the simple operations must include changes of distribution of observed squares. The new observed squares must be immediately recognisable by the computer. I think it is reasonable to suppose that they can only be squares whose distance from the closest of the immediately previously observed squares does not exceed a certain fixed amount. Let us say that each of the new observed squares is within L squares of an immediately previously observed square.

In connection with "immediate recognisability", it may be thought that there are other kinds of square which are immediately recognisable. In particular, squares marked by special symbols might be taken as imme-

[251]

diately recognisable. Now if these squares are marked only by single symbols there can be only a finite number of them, and we should not upset our theory by adjoining these marked squares to the observed squares. If, on the other hand, they are marked by a sequence of symbols, we cannot regard the process of recognition as a simple process. This is a fundamental point and should be illustrated. In most mathematical papers the equations and theorems are numbered. Normally the numbers do not go beyond (say) 1000. It is, therefore, possible to recognise a theorem at a glance by its number. But if the

paper was very long, we might reach Theorem 157767733443477; then, further on in the paper, we might find "... hence (applying Theorem 157767733443477) we have ...". In order to make sure which was the relevant theorem we should have to compare the two numbers figure by figure, possibly ticking the figures off in pencil to make sure of their not being counted twice. If in spite of this it is still thought that there are other "immediately recognisable" squares, it does not upset my contention so long as these squares can be found by some process of which my type of machine is capable. This idea is developed in III below.

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]*L*[illegible]

[illegible]“[illegible]”[illegible]1000[illegible]157767733443477[illegible]“……[illegible]157767733443477[illegible]……”[illegible]“[illegible]”[illegible]III[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

The simple operations must therefore include:

(*a*) Changes of the symbol on one of the observed squares.

(*b*) Changes of one of the squares observed to another square within *L* squares of one of the previously observed squares.

It may be that some of these changes necessarily involve a change of state of mind. The most general single operation must therefore be taken to be one of the following:

(*A*) A possible change (*a*) of symbol together with a possible change of state of mind.

(*B*) A possible change (*b*) of observed squares, together with a possible change of state of mind.

The operation actually performed is determined, as has been suggested on p. 250, by the state of mind of the computer and the observed symbols. In particular, they determine the state of mind of the computer after the operation is carried out.

[illegible]

(a) [illegible]

(b) [illegible]*L*[illegible]

[illegible]

(A) [illegible](a)[illegible]

(B) [illegible](b)[illegible]

[illegible]250[illegible]

[illegible]

We may now construct a machine to do the work of this computer. To each state of mind of the computer corresponds an "*m*-configuration" of the machine. The machine scans *B* squares corresponding to the *B* squares observed by the computer. In any move the machine can change a symbol on a scanned square or can change any one of the scanned squares to another square distant not more than *L* squares from one of the other scanned

[252]

squares. The move which is done, and the succeeding configuration, are determined by the scanned symbol and the *m*-configuration. The machines just described do not differ very essentially from computing machines as defined in § 2, and corresponding to any machine of this type a computing machine can be constructed to compute the same sequence, that is to say the sequence computed by the computer.

[illegible] m- [illegible] *B* [illegible] *B* [illegible] *L* [illegible] m- [illegible] §2 [illegible]

[illegible]

[illegible]

[illegible] 1936 [illegible] 1950 [illegible] 10 [illegible] *Mind* [illegible]

“[illegible]?” [illegible]

[illegible]

[illegible]

> “[illegible]?” [illegible] [4]

[illegible] 2 [illegible]

[illegible]

> [illegible]·[illegible]20[illegible]40[illegible]
> [illegible]
> [illegible]
> [illegible][5]

[illegible]56[illegible]1968[illegible]2001[illegible]
[illegible]

1950[illegible]10[illegible]
[illegible]

1951[illegible]3[illegible]15[illegible]·[illegible]
[illegible]·[illegible]·[illegible]BBC[illegible]“[illegible]”[illegible]
[illegible]

1951[illegible]12[illegible]
[illegible]·[illegible]
[illegible]

1952[illegible]1[illegible]·[illegible]
[illegible]

1952[illegible]2[illegible]7[illegible]·[illegible]
[illegible]1885[illegible]11
[illegible]

> [illegible]
> [illegible]2[illegible][6]

[illegible]“[illegible]”[illegible]
[illegible]“[illegible]”[illegible]

[illegible]11[illegible]1885[illegible]“[illegible]
[illegible]”[illegible]13[illegible]16
[illegible]

[illegible]·[illegible]·[illegible]1849—
1912[illegible]13[illegible]
[illegible]

[illegible]11[illegible]Henry
Labouchere[illegible]1885[illegible]8[illegible]6[illegible]

[illegible]11[illegible][7]

[illegible]11[illegible]“[illegible]”[illegible]“[illegible]”[illegible][8]

[illegible]11[illegible]·[illegible]1854—1900[illegible]1895[illegible]

[illegible]20[illegible]50[illegible]1[illegible]

[illegible]20[illegible]40[illegible][9][illegible]“[illegible]”[illegible]

[illegible]20[illegible]50[illegible]50[illegible]“[illegible]”[illegible]“20[illegible]50[illegible]60[illegible]1000[illegible]”[illegible][10]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible][11][illegible][12]

[illegible]1951[illegible]“[illegible]”[illegible][13]

[illegible]·[illegible]1952[illegible]“[illegible]”[illegible][14]

[illegible]

[illegible]·[illegible]-[illegible]1953[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]1953[illegible]1954[illegible][15]

[illegible]

[illegible]

[illegible]·[illegible]

[illegible]

[illegible][16]

[illegible]1954[illegible]6[illegible]7[illegible]

[illegible]41[illegible]

---

[1] □□□·□□□□*Alan Turing*□*The Enigma*□Simon & Schuster□1983□□104□

[2] □□·□□□□“Why Gödel Didn’t Have Church’s Thesis”□*Information and Control*□Vol. 54□Nos. 1/2□□July/Aug .1982□□14□

[3] □□□·□□□□*Collected Works*□*Volume II: Publications 1938-1974*□Oxford University Press□1990□□306□Judson C. Webb□□□□292□□□□□□297□□Oron Shagrir□“Gödel on Turing on Computability”□http://edelstein.huji.ac.il/staff/shagrir/papers/Goedel_on_Turing_on_Computability.pdf□

[4] □□·□□□“Computing Machinery and Intelligence”□*Mind*□Vol. LIX□No. 236□October 1950□□442□

[5] Arthur C. Clark□*2001: A Space Odyssey*□New American Library□1968□□ch. 16□

[6] □□□□http://www.swarb.co.uk/acts/1885Criminal_Law_AmendmentAct.shtml□2008□4□□□□□

[7] H. Montgomery Hyde□*The Love That Dared Not Speak Its Name: A Candid History of Homosexuality in Britain*□Little□Brown and Company□1970□□134□□□□□□□□*The Other Love*□□□□□

[8] □□□136□

[9] □□□□*Alan Turing*□467-471□

[10] David K. Johnson□*The Lavender Scare: The Cold War Persecution of Gays and Lesbians in the Federal Government*□University of Chicago Press□2004□□76□

[11] Johnson□*Lavender Scare*□7-8□

[12] Johnson□*Lavender Scare*□108-109□

[13] Johnson□*Lavender Scare*□133□

[14] □□□□*Alan Turing*□474□

[15] Hyde□*The Love That Dared Not Speak Its Name*□214-216□

[16] □□·□□□*A Madman Dreams of Turing Machines*□Alfred A. Knopf□2006□□214□

[illegible]12[illegible]

[illegible]

1958[illegible]IBM[illegible]IBM 704[illegible]IBM[illegible]·[illegible]·[illegible]·[illegible]*11.26[illegible]

$$^{*}11\cdot 26.\vdash :.(\exists x):(y).\phi(x,y):\supset:(y):(\exists x).\phi(x,y)$$

[illegible]

11*26/EXAYGXY-AYEXGXY

[illegible]*1[illegible]*5[illegible]220[illegible]*9[illegible]*13[illegible]158[illegible][1]

[illegible][2]1954[illegible]·[illegible]Presburger[illegible][3]

1957[illegible]·[illegible]·[illegible]·[illegible]“[illegible]”[illegible]JOHNNIAC[illegible]·[illegible]·[illegible][4][illegible]“[illegible]”[illegible]

[illegible]·[illegible]“[illegible]”[5]

[illegible]2000[illegible]20[illegible]23[illegible][6][illegible]·[illegible]·[illegible]1952[illegible]1967[illegible][7][illegible]1913[illegible]

[illegible]“[illegible]”[illegible]

[illegible]·[illegible]

[illegible]·[illegible]1917[illegible]1918[illegible]1928[illegible]·[illegible]·[illegible]120[illegible]——[illegible]——[illegible]

[illegible]·[illegible]·[illegible]1934[illegible]1939[illegible]

[illegible]·[illegible]·[illegible]·[illegible]·[illegible]

[illegible]Ausssagenkalkul[illegible]

[illegible]engere Funktionenkalkul[illegible]1938[illegible]engere Pradilatenkalkul[illegible]erweiterte Funktionenkalkül[illegible]erweiterte Prädikatenkalkül[illegible]

[illegible]

[illegible]

7[illegible]

[illegible]

[illegible]

10[illegible]

[illegible]

[illegible]A[illegible]B[illegible]C[illegible]X[illegible]Y[illegible]Z[illegible]

[illegible]

[illegible]v[illegible]vel[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]aut[illegible]X[illegible]Y[illegible]

$$X \vee Y$$

[illegible]X[illegible]Y[illegible]

X	**Y**	**X ∨ Y**
[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]

□	□	□
□	□	□

□□□□□□□□□□□□□□□□□□v□□□XY□□□$X \vee Y$□

□□□

□□

□□□□□□□□□□□□□□□□aq.□□□□□□□äquivalent□□eq.□□□□□□equivalent□□□□□□

$$X \vee Y \text{ eq. } XY$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

“□”□□□□□□&□□□□□

$$X \,\&\, Y$$

□□□□□□X□Y□□□□□□□□□□□□

X	Y	$X \,\&\, Y$
□	□	□
□	□	□
□	□	□
□	□	□

“□”□□□□□□□□conjuction□□□□□□□□and□□□□□□□□□□“□”□□□□□□□disjuncton□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□

$$X \vee X \text{ eq. } X$$

$$X \,\&\, X \text{ eq. } X$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□v□□□□□□&□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$X \,\&\, Y \vee Z \text{ eq. } X \,\&\, (Y \vee Z)$$

□□□□□□□□□□

$$(X \,\&\, Y) \vee Z$$

□□□□X□□□Y□□□Z□□□□□□□□□□□□□□□□□□□□□□

□□□well-formed formula□□□□wff□□□□woofs□□□□□□□□true□

false□T□F□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□T□□□□
□□□□□□F□□□□□□□□□□□□□□□□□□□□□T□F□

□□□□□□□□□□□□□

$$X \vee T \text{ eq. } T$$
$$X \vee F \text{ eq. } X$$
$$X \,\&\, T \text{ eq. } X$$
$$X \,\&\, F \text{ eq. } F$$

□□□□□□□□□□□□□□□□□□□□□

$$X \vee Y \text{ eq. } Y \vee X$$
$$X \,\&\, Y \text{ eq. } Y \,\&\, X$$

□□□□□□□□□□

$$X \vee (Y \vee Z) \text{ eq. } (X \vee Y) \vee Z$$
$$X \,\&\, (Y \,\&\, Z) \text{ eq. } (X \,\&\, Y) \,\&\, Z$$

□□□□□□□□□□

$$X \vee (Y \,\&\, Z) \text{ eq. } (X \vee Y) \,\&\, (X \vee Z)$$
$$X \,\&\, (Y \vee Z) \text{ eq. } (X \,\&\, Y) \vee (X \,\&\, Z)$$

□□□□□□F□□□0□□T□□□1□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□“□□□”□□□□□“□□□”□□□□□□□□□□□□□□□□□□
□□□□

□□□□□□□□□□□□□□□□□“□”□“□”□□□□□□□□□□□□

X	$-X$
F	T
T	F

□□□□□□□□□□□□□□□□□□□□□□□“□”□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□

$$--X \text{ eq. } X$$

□□□□□□□□□□□□

$$X \vee -X \text{ eq. } T$$
$$X \,\&\, -X \text{ eq. } F$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□19□□□□□□□□·□·□
□□1806—1871□□□□□□□□□·□□□□□□□□□□□□□□□□□□□□□□□

$$-(X \vee Y) \text{ eq. } -X \,\&\, -Y$$
$$-(X \,\&\, Y) \text{ eq. } -X \vee -Y$$

□□□□□□□□□□□□□□□□□□□□□□“□□□□□□□”□□□□□($X \vee Y$)□□
□□□□“□□□□□□□□□□”□□□□□□-X & -Y□□□□□□□“□□□□□□□□

□□”□□□- (X & Y)□□□□□□□□□□□□“□□□□□□□□□□□□□□□□□□□□□□……”□□□□-X v -Y□

□□□□□□□·□□□□□□□□□□□□□□□□□□□□□□□□□□

$$X \text{ v } Y \text{ eq. } -(-X \,\&\, -Y)$$

$$X \,\&\, Y \text{ eq. } -(-X \text{ v } -Y)$$

□“□”□“□”□□□□□□□□□“□”□□□“□”□“□”□□□“□”□□□□□□□□□□□□□□□□□□□□“□□□□”□□□□□□□□□□□□□□□□□

$$X \,\&\, -Y \text{ v } Z$$

□□□□□□□□□□□□□□□□□□□□□□&□v□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$-X \text{ v } (Y \,\&\, -Z)$$

□□

X	***Y***	***Z***	***X*** &-***Y*** v ***Z***	-***X*** v (***Y*** &-***Z***)
F	F	F	F	T
F	F	T	F	T
F	T	F	F	T
F	T	T	F	T
T	F	F	T	F
T	F	T	T	F
T	T	F	F	T
T	T	T	T	F

□□□□□□□□□□□□□□

$$X \,\&\, -Y \text{ v } Z \text{ eq. } -□-X \text{ v } (Y \,\&\, -Z)□$$

□□□□□“□□”□□□□□□□□□□□□□□□□□□□□□□□

$$(X \text{ v } Y) \,\&\, -(X \,\&\, Y)$$

□□□□□□□□□□□□□□□□□□□□□□

X	***Y***	***X*** v ***Y***	***X*** & ***Y***	(***X*** v ***Y***)& - (***X*** & ***Y***)
F	F	F	F	F
F	T	T	F	T
T	F	T	F	T
T	T	T	T	F

[illegible] X [illegible] Y [illegible] “[illegible]” [illegible]

[illegible]·[illegible] “[illegible]” [illegible]

$$(X \vee Y) \,\&\, (-X \vee -Y)$$

[illegible]

[illegible] [8]

[illegible] $X \rightarrow Y$ [illegible] “X [illegible] Y” [illegible] “[illegible] X [illegible] Y [illegible]” [illegible] “[illegible]” [illegible]

X	***Y***	***X* → *Y***
F	F	T
F	T	T
T	F	F
T	T	T

[illegible] X [illegible] Y [illegible] $X \rightarrow Y$ [illegible] $X \rightarrow Y$ [illegible] $X \rightarrow Y$ [illegible] X [illegible] Y [illegible] X [illegible] Y [illegible] Y [illegible] X [illegible] Y [illegible] $X \rightarrow Y$ [illegible]

[illegible] “[illegible]” [illegible]

[illegible]

[illegible]

$$X \rightarrow Y \text{ eq. } -Y \rightarrow -X$$

[illegible]

$$X \rightarrow Y \text{ eq. } -X \vee Y$$

[illegible] X [illegible] Y [illegible] $X \rightarrow Y$ [illegible]

$$X \rightarrow Y \text{ eq. } -(X \,\&\, -Y)$$

[illegible]

$$X \,\&\, Y \rightarrow X$$

$$X \,\&\, Y \rightarrow Y$$

[illegible] “[illegible]” [illegible]

X	***Y***	***X* ~ *Y***
F	F	T
F	T	F

T	F	F
T	T	T

$X \sim Y$[illegible]X[illegible]Y[illegible]

$$X \sim Y \text{ eq. } (X\ Y)\&(Y \rightarrow X)$$

[illegible]T→T[illegible]F→F[illegible]T→F[illegible]F→T[illegible]T→F[illegible]F→T[illegible]

[illegible]

$$X \vee Y \vee (-X \& -Y)$$

[illegible]X[illegible]Y[illegible]X[illegible]Y[illegible]

X	***Y***	***X* v *Y* v (-*X* & -*Y*)**
F	F	T
F	T	T
T	F	T
T	T	T

[illegible]X[illegible]Y[illegible]

[illegible]

X	***Y***	***X* & *Y* & -*X***
F	F	F
F	T	F
T	F	F
T	T	F

[illegible]

[illegible]

X	***Y***	***X* v (*Y* & -*Y*)**
F	F	F

F	T	F
T	F	T
T	T	T

[illegible]*X*[illegible]*Y*[illegible]

[illegible]

[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]

[illegible]

[illegible]100[illegible]2^{100}[illegible]1267650600228229401496703205376[illegible]10^{30}[illegible]38[illegible]3000[illegible]

[illegible]55[illegible]

[illegible]

[illegible]“[illegible]……”[illegible]·[illegible]“[illegible]……”[9][illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]

[illegible]IsPrime[illegible]IsPrime[illegible]IsPrime(7)[illegible]IsPrime(9)[illegible]IsPrime(*x*)[illegible]

[illegible]*x*[illegible]*y*[illegible]Loves(*x*, *y*)[illegible]Loves(Pat, Terry)[illegible]“Pat[illegible]Terry”[illegible]Loves(Pat, Terry)[illegible]Loves(Terry, Pat)[illegible]

[illegible]

Loves(Pat, Terry) & Loves(Terry, Pat)

[illegible]

Loves(Pat, Terry) ~ Loves(Terry, Pat)

[illegible]

[illegible]

Loves(*x,* Pat)

[illegible]Pat[illegible]Pat[illegible]

[illegible]quantifier[illegible]quantors[illegible]

(*x*)Loves(*x*, Pat)

[illegible]*x*[illegible]“[illegible]*x*”[illegible]“[illegible]*x*[illegible]*x*[illegible]Pat”[illegible]Pat[illegible]Pat[illegible]

(*x*)Loves(Pat, *x*)

[illegible]

[illegible]“[illegible]”[illegible]E[illegible]∃[illegible]

(∃*x*)Loves(*x,* Terry)

[illegible]“[illegible]*x*[illegible]*x*[illegible]Terry”[illegible]Terry[illegible]Terry[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]Pat[illegible]Terry[illegible]Kim[illegible]

(*x*)Loves(Kim, *x*)

[illegible]

Loves(Kim, Pat) & Loves(Kim, Terry) & Loves(Kim, Kim)

[illegible]

(∃*x*)Loves(Kim, *x*)

[illegible]

Loves(Kim, Pat) v Loves(Kim, Terry) v Loves(Kim, Kim)

[illegible]

[illegible]·[illegible]Terry[illegible]

- (*x*)Loves(*x,* Terry) eq. (∃*x*)-Loves(*x,* Terry)

[illegible]Terry[illegible]

(*x*) -Loves(*x,* Terry) eq. - (∃*x*) Loves(*x,* Terry)

□□□

$$(x)\ \text{Loves}(x, \text{Terry})\ \text{eq.}\ -\ (\exists x)\ -\text{Loves}(x, \text{Terry})$$

□□□□□□□□□“□□□□□□Terry□□□□”□□□□

$$(\exists x)\ \text{Loves}(x, \text{Terry})\ \text{eq.}\ -\ (x)\ -\text{Loves}(x, \text{Terry})$$

□□□□□□□□□Terry□□□□

□□□□□□·□□□·□□□1839—1914□□□□□□□□□□□□□□□Σ□□□□□□□□□□□□□□□□□□□□Π□□□

□□□□□□□□□x□□□x□□□□□y□□□□□

$$(\exists x)\text{Loves}(x, y)$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□x□□z□

$$(\exists z)\text{Loves}(z, y)$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□y□□□y□□□□□□y□□□□□□□□□□□□□□□

□□“□□”□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□x□□□□□□□□□□□□□□□□

$$(x)[\text{Loves}(x, \text{Kim})\ \&\ \text{Loves}(x, \text{Pat})]$$

□□Kim□Pat□□□

$$(x)\text{Loves}(x, \text{Kim})\ \&\ (x)\ \text{Loves}(x, \text{Pat})$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$(y)\ \text{Loves}(y, \text{Kim})\ \&\ (x)\ \text{Loves}(x, \text{Pat})$$

□□□□□□Kim□□Pat□□□□

$$(\exists x)[\ \text{Loves}(x, \text{Kim})\ \&\ \text{Loves}(x, \text{Pat})]$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$(\exists x)\ \text{Loves}(x, \text{Kim})\ \&\ (\exists x)\ \text{Loves}(x, \text{Pat})$$

□□□□□□□□Kim□□□□□□Pat□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□

$$(x)[\ \text{Loves}(x, \text{Kim})\ \text{v}\ \text{Loves}(x, \text{Pat})]$$

□□□□□□□Kim□□□□Pat□□□□□Kim□□Pat□□□□□□□□□□□□□Terry□Kim□□□Pat□□□Terry□Pat□□□Kim□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$(x)\ \text{Loves}(x, \text{Kim})\ \text{v}\ (x)\ \text{Loves}(x, \text{Pat})$$

□□□□□□□Kim□□□□□□Pat□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□

$$(\exists x)[\ \text{Loves}(x, \text{Kim})\ \text{v}\ \text{Loves}(x, \text{Pat})]$$

□□□□□□□□Kim□□□Pat□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$(\exists x)\text{Loves}(x, \text{Kim}) \vee (\exists x)\ \text{Loves}(x, \text{Pat})$$

□□□□□□□□□□□□□□□□□□□□□□□□□*A*□□□*a*□□□□□□□□□□□□□□□□

$$(x)A(x) \rightarrow A(a)$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$A(a) \rightarrow (\exists x)A(x)$$

□□□□□□□□□□□□□

$$(\exists x)(y)\text{Loves}(x, y)$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$(\exists x)[(y)\text{Loves}(x, y)]$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$(y)\ (\exists x)\ \text{Loves}(x, y)$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□Kim□Kim□Terry□
Terry□□□□□□□□□□□□

$$(y)\ (\exists x)\ \text{Loves}(x, y)$$

□□□□□□

$$(\exists x)(y)\text{Loves}(x, y)$$

□□□
□□□□□□□*11.26□□□□□□□□□□□

$$*11\cdot 26. \vdash :.(\exists x):(y).\phi(x,y) :\supset: (y):(\exists x).\phi(x,y)$$

□□*ø*(*x*, *y*)□□□□□□□□□□□□□

$$(\exists x)(y)\phi(x,\ y) \rightarrow (y)(\exists x)\phi(x,\ y)$$

□□□
□□□
□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$(x)[F(x) \vee -F(x)]$$

□□□□□□□□□□□□□□□□*F*□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$(\exists x)(F(x)\ \&\ -F(x))$$

□□

$$(x)F(x)$$

□□□□*x*□□□□□□□□□□□*F*□□□□□□□□□□□□□□□□□□□□□□□□*F*□□“□□□□□□”□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□*F*□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□

□□□
□□□□□□□$\mathfrak{A}$□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathfrak{A}$□□□□□□□□□□□□□-$\mathfrak{A}$□□□□□□□□□□
□□□□□□□$\mathfrak{A}$□□□□□□□□□-$\mathfrak{A}$□□□□□□□

□□
□□□□□□□

□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□“□□□□”□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

(a) $X \vee X \rightarrow X$

(b) $X \rightarrow X \vee Y$

(c) $X \vee Y \rightarrow Y \vee X$

(d) $(X \rightarrow Y) \rightarrow (Z \vee X \rightarrow Z \vee Y)$

□□□□□□□□□□□□□□□□□□□□ X & Y □□□ - (-X v -Y)□□□□□□□□□□□□□□
□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□ F □□□□□□□□□□□□□□

(e) $(x)F(x) \rightarrow F(y)$

(f) $F(y) \rightarrow (\exists x)\, F(x)$

□□□□□□□□□□□□□□□□□□□□□□□□□

1. □□□
□□□□□□□□□□□□□□□□□□□□□□□□□□

2. □□□□□□□ $\mathfrak{A}$ □□□□□□□ $\mathfrak{A} \rightarrow \mathfrak{B}$ □□□□ $\mathfrak{B}$ □□□

□□□□□□□□□□□modus ponens□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□·□□□□□□□□□□□□□□□□□□[11]

□□
□□□□□□□□□□□□□□□□□□□□

□□□□□ $\mathfrak{A}$ □ $\mathfrak{B}$ □□□□□□□□□(c)□□□(1)□□□□□□

$$\mathfrak{A} \vee \mathfrak{B} \rightarrow \mathfrak{B} \vee \mathfrak{A}$$

□□□□□□□□□□□□□□

□□□
□□□
□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□·□□□□1854—
1912□□□□□□□□□□□□□□□“□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□”□[12]

□□
□□□□□□□□□□□□□□□□□□□□□

□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□·□□□□1929□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□1930□□□□□□□□□□□□□□□□□□□
□□□□□□□□

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]——[illegible]

[illegible]𝔄[illegible]𝔄[illegible]𝔄[illegible]

[illegible]𝔄[illegible]𝔄[illegible]𝔄[illegible]𝔄[illegible]𝔄[illegible]1930[illegible]

1931[illegible]——[illegible]“[illegible]”[illegible]Bew[13][illegible]

[illegible]“[illegible]”[illegible]I[illegible][14][illegible]

[illegible]1930[illegible]1931[illegible]

[illegible]—[illegible]—[illegible]“[illegible]”[illegible]·[illegible]

> [illegible]“[illegible]”[illegible][15]

[illegible]·[illegible]

[illegible]9[illegible]“[illegible]”[illegible]9[illegible]“[illegible]”[illegible]249[illegible]176[illegible]

[illegible]I[illegible]“[illegible](a)”[illegible]II[illegible]“[illegible](b)”[illegible]“[illegible](b)“[illegible]”[illegible]”[illegible]

[illegible]

II. [Type (*b*)].

If the notation of the Hilbert functional calculus† is modified so as to be systematic, and so as to involve only a finite number of symbols, it becomes possible to construct an automatic‡ machine, $\mathcal{K}$, which will find all the provable formulae of the calculus§.

---

† The expression "the functional calculus" is used throughout to mean the *restricted* Hilbert functional calculus.

‡ It is most natural to construct first a choice machine (§ 2) to do this. But it is then easy to construct the required automatic machine. We can suppose that the choices are always choices between two possibilities 0 and 1. Each proof will then be determined by a sequence of choices $i_1, i_2, \ldots, i_n$ ($i_1 = 0$ or 1, $i_2 = 0$ or 1, …, $i_n = 0$ or 1), and hence the number $2^n + i_1 2^{n-1} + i_2 2^{n-2} + \ldots + i_n$ completely determines the proof. The automatic machine carries out successively proof 1, proof 2, proof 3, …

§ The author has found a description of such a machine.

II. [[illegible](b)]

[illegible]†[illegible]‡[illegible]$\mathcal{K}$[illegible]§[illegible]

---

† "[illegible]"[illegible]

‡ [illegible]§2[illegible]0[illegible]1[illegible]$i_1, i_2, \ldots, i_n$[illegible]$i_1 = 0$[illegible]1, $i_2=0$[illegible]1,…, $i_n=0$[illegible]1[illegible]$2^n + i_1 2^{n-1} + i_2 2^{n-2} + \ldots + i_n$[illegible]1[illegible]2[illegible]3……

§ [illegible]

[illegible]$\mathcal{K}$[illegible]Kalküi[illegible]$\mathcal{K}$[illegible]

[illegible]"[illegible]"[illegible]

$$(x)(\exists y)\phi(x,\ y)$$

$$(y)(\exists x)\phi(y,\ x)$$

$$(x)(\exists y)(\phi(x,\ y))$$

[illegible] x_i [illegible] 1878—1956 [illegible]

$$(A \vee B)\ \&\ (C \vee D)$$

[illegible]

$$\&{\vee}AB{\vee}CD$$

[illegible]

[illegible] 1936 [illegible] $\mathcal{K}$ [illegible] 10 [illegible] 7 [illegible] $\mathcal{K}$ [illegible]

[illegible] —— [illegible] “[illegible]……” [illegible] n [illegible]

[illegible] n [illegible] 2^n [illegible] 0 [illegible] 1 [illegible] n [illegible] 2^n [illegible] 0 [illegible] n-1 [illegible] n [illegible]

[illegible] $\mathcal{K}$ [illegible]

[illegible] 1889 [illegible] · [illegible] [15] [illegible] · [illegible] 1888 [illegible] [16] [illegible]

[illegible] “[illegible]” [illegible] “[illegible]” [illegible] 12 [illegible] 13 [illegible] 1 [illegible] 0 [illegible]

[illegible]

1. [illegible]
2. [illegible]
3. [illegible]

4. □□□□□□□□□□□□□

5. □□

□5□□

□□□

□□□□□□□□□□□□□□□□□□□□0, 1, 2, 3,…□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□12□□□□13□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□“□□□□”□□□□

□□□□□□□□1934□□□□□□□□□□□□□□□□□□□□□□·□□□□“ ″”□□□□□□□□□□□□□□□□□□a□□□□□□$a´$□□□□□□□□$a″$□□□□□□□□□□□□□□□0□□□□□□□$0´$□0□□□$0″$□□□□□□□[17]□□□□□□□□□□□□□□□0□“ ″”□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□·□□□□1927□□□□□□□□□[18]

□□□□□□□□□□□□□□□□□□□□□□□□□0□□□□□□□u□□□□□□□□□□□□□□□u□□0□□1□□□□□□□□□□□□□□□□□□u□□0□□u□□□□□$u´$□$u″$□$u‴$□□□□□□□□□□□□□□□□□□□□□□□□□□□n□x□□□□□□□□□□□□□□□□□□□□□□□$u^{(r)}$□□u□□□□□r□“ ″”□□□$u⁗$□□□□$u^{(5)}$□□□□□□□□□□□□□□□□

□□□□□□□□□□N(x)□□x□□

□□□□□□□□□□□F(x, y)□□y□x□□□□□□□□□□□□□y = x + 1□□□□□□□□□□□□F□□□□□□□□□□□□□□□□□□□□□□□□□□□□□y□□x□□□□□□□□□□

□□□□□□□□□□□□□□□□□□Succ□□□Sum(x, y, z)□□□□□z□□x + y□□□□□□□□□□□□□□□□□□Sum□□□□□□□

(x)Sum(x, u, x)

(x)Sum(u, x, x)

(x)(y)(z)(r)(s)(Sum(x, y, z)&Succ(y, r)&Succ(z, s)→Sum(x, r, s))

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□x + y = z□r = y + 1□s = z + 1□□□x + r = s□

□□□□□□□□□Product(x, y, z)□□□

(x) Product (x, u, u)

(x) Product (u, x, u)

(x)(y)(z)(r)(s)(Product (x, y, z) & Succ(y, r) & Sum(z, x, s) → Product (x, r, s))

□□□□□□□□□□□□□□□□□□□□□□□□□□$x \times y = z$□$r = y + 1$□$z + x = s$□□□$x \times r = s$□

□□□□□□□□□□□□□□□□□□□IsEven(x)□

$$((\exists y)\,\mathrm{Product}(y, u'', x)$$

□□□□□□□y□□$x = y \times 2$□□□IsEven(x)□□□IsOdd(x)□-IsEven(x)□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

Now let α be a sequence, and let us denote by $G_\alpha(x)$ the proposition "The x-th figure of α is 1", so that[II] — $G_\alpha(x)$ means "The x-th figure of α is 0".

[II] The negation sign is written before an expression and not over it.

□α□□□□□□□□□$G_\alpha(x)$□□□□"α□□x□□□□1"□□□□-$G_\alpha(x)$[II]□□"α□□x□□□□0"□

[II] □□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□α□□□1□0□□□□□□□□□□□□□□□□□□6□□□□□□□□2□□□□□□□

1011011010…

□□□

□□□0□□□□□□□□□□□□□□□□$G_\alpha(0)$□□□□□$G_\alpha(1)$□□□□□$G_\alpha(2)$□□□□□$G_\alpha(3)$□□□□□$G_\alpha(4)$□□□□□□□□□□□□□□□□□□□□□□l□□□□□□□□□□□□□□□

0101010101…

□□□□□□□□□□□□1/3□□□□□□□□□□□□$G_\alpha(x)$□□□IsOdd(x)□□□□□□□□□□□□□□□□□□□□□

Suppose further that we can find a set of properties which define the sequence α and which can be expressed in terms of $G_\alpha(x)$ and of the propositional functions $N(x)$ meaning "x is a non-negative integer" and $F(x, y)$ meaning "$y = x + 1$".

□□□□□□□□□□□□□□□□□□□□□α□□□□□□$G_\alpha(x)$□□□□□$N(x)$□$F(x, y)$□□□□□□□□□□□□$N(x)$□□"x□□□□□□□"□$F(x, y)$□□$y = x + 1$□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□F□□□N□

□□□□□□□□□"□□□□"□□□□□□□□□□□□□□□□□□□$G_\alpha(x)$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□Sum□Product□□□□

When we join all these formulae together conjunctively, we shall have a formula, $\mathfrak{A}$ say, which defines *α.*

□□□□□□□□□□□□□□□□□□□□□□□□□□□□*α*□□□□□□□□□□□$\mathfrak{A}$□

□□□□□□□□□□□□□□□□□□*α*□□□□□□□□□□*α*□□□□□

The terms of $\mathfrak{A}$ must include the necessary parts of the Peano axioms, viz.,

$$(\exists u)\ N(u)\ \&\ (x)\ (N(x) \rightarrow (\exists y)\ F(x, y))\ \&\ (F(x, y) \rightarrow N(y)),$$

which we will abbreviate to *P.*

$\mathfrak{A}$□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$(\exists u)N(u)\&(x)(N(x)\rightarrow(\exists y)F(x,y))\&(F(x,y)\rightarrow N(y))$$

□□□□□□□□□□□□*P*□

□□□*P*□□□□□□□□□□□□□□□□□□□□□□□□□□□*u*□□□□□□□□□□□□□□□□□□*x*□□□□□□*y*□□□□ □□*S*[19]□□□□□□□□□□□□□□□

$$(x)(\exists y)S(x, y)$$

$$(\exists x)(y) - S(y, x)$$

$$(x)(y)(r)(s)(S(x, r)\ \&\ S(y, r)\ \&\ S(s, x) \rightarrow S(s, y))$$

□□□*r*□*x*□*y*□□□□ *x*□*s*□□□□□□□*y*□□*s*□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

When we say "$\mathfrak{A}$ defines *α*", we mean that — $\mathfrak{A}$ is not a provable formula, and also that, for each *n*, one of the following formulae (A_n) or

(B_n) is provable.

$$\mathfrak{A}\ \&\ F^{(n)} \rightarrow G_\alpha(u^{(n)}), \qquad (A_n)¶$$

$$\mathfrak{A}\ \&\ F^{(n)} \rightarrow \left(-G_\alpha(u^{(n)})\right), \qquad (B_n),$$

where $F^{(n)}$ stands for $F(u, u') \ \&\ F(u', u'')\ \&\ \ldots\ F(u^{(n-1)}, u^{(n)})$.

¶ A sequence of *r* primes is denoted by $^{(r)}$.

□□□□"$\mathfrak{A}$□□□*α*"□□□□□-$\mathfrak{A}$□□□□□□□□□□□□□□□□□□□□□*n*□□□□□□(A_n)□(B_n)□ □□□□□□□□□□

$$\mathfrak{A}\ \&\ F^{(n)} \rightarrow G_\alpha(u^{(n)}), \qquad (A_n)¶$$

$$\mathfrak{A}\ \&\ F^{(n)} \rightarrow \left(-G_\alpha(u^{(n)})\right), \qquad (B_n)$$

□□$F^{(n)}$□□$F(u,u')$&$F(u',u'')$&…$F(u^{(n-1)},u^{(n)})$□

¶ □□r□“ ′ ”□□□□$^{(r)}$□

□□(A_n)□□□□□□□□□□□□□□□□□□□□□□□□□□“ ″”□□□□□□□F□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□1□0□□□□2□1□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$F(u', u'''')$□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□-$F(u', u'')$□-$F(u',$
$u)$□□□$u^{(n)}$□□□□□$F^{(n)}$□□□□□□□□□

□□n□□□□□□□0□□□□□□0□□1□□2□□……□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□$F^{(n)}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□0
□□□□□□u□□□□□□□□□□□IsOdd□□□□□u''□□□□□□□F□□□□□2□n□□□□
□□□

□□□□□□□□□□□□□□□□□□□□□□□□□

$$B_0 : \mathfrak{A} \,\&\, F^{(2)} \rightarrow -\,\mathrm{IsOdd}(u)$$

$$A_1 : \mathfrak{A} \,\&\, F^{(2)} \rightarrow \mathrm{IsOdd}(u')$$

$$B_2 : \mathfrak{A} \,\&\, F^{(2)} \rightarrow -\,\mathrm{IsOdd}(u'')$$

$$A_3 : \mathfrak{A} \,\&\, F^{(3)} \rightarrow \mathrm{IsOdd}(u''')$$

$$B_4 : \mathfrak{A} \,\&\, F^{(4)} \rightarrow -\,\mathrm{IsOdd}(u'''')$$

$$A_5 : \mathfrak{A} \,\&\, F^{(5)} \rightarrow \mathrm{IsOdd}(u''''')$$

□□□□□□□□□□$\mathfrak{A}$□□□□IsOdd□□□□□□□□□□□□□□□□□□□□□□□□□□□0□1□0□
1□0□1□□□□□1□□□□□A_n□0□□B_n□□□□□

[253]

I say that α is then a computable sequence: a machine $\mathcal{K}_\alpha$ to compute α can be obtained by a fairly simple modification of $\mathcal{K}$.

□□□□□□α□□□□□□□□□□□□□□$\mathcal{K}$□□□□□□□□□□□α□□□$\mathcal{K}_\alpha$□

□□□□□□□□□$\mathcal{K}$□□□□□□□□□□□□□□□□□□□□□□

We divide the motion of $\mathcal{K}_\alpha$ into sections. The n-th section is devoted to finding the n-th figure of α. After the (n - 1)-th section is finished a double colon :: is printed after all the symbols, and the succeeding work is done wholly on the squares to the right of this double colon. The first step is to

write the letter “*A*” followed by the formula (A_n) and then “*B*” followed by (B_n).

□□□$\mathcal{K}_\alpha$□□□□□□□□□□n□□□□□□□□□α□□n□□□□□(n-1)□□□□□□□□□□□□□□□□□□□□□□□□□::□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□“A”□□□□□□□A_n□□□□□“B”□□□□□□B_n□

□□□□□□n□□□5□□□□□□“A”□“B”□□□□□□□□□□□□□□□□□□□□

$$\mathbf{A}\,\mathfrak{A}\ \&\ F^{(5)} \rightarrow \mathrm{IsOdd}(u'''''')\quad \mathbf{B}\,\mathfrak{A}\ \&\ F^{(5)} \rightarrow - \mathrm{IsOdd}(u'''''')$$

□□□□□□□□□□□□□□□□□□“A”□“B”□□□□□□□$\mathfrak{A}$□□□□□□□□□□□□□□□$F^{(5)}$□□□□□□□□□□□□□IsOdd□□□□□□□□Product□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□

The machine $\mathcal{K}_\alpha$ then starts to do the work of $\mathcal{K}$, but whenever a provable formula is found, this formula is compared with (A_n) and with (B_n). If it is the same formula as (A_n), then the figure “1” is printed, and the *n*-th section is finished. If it is (B_n), then “0” is printed and the section is finished. If it is different from both, then the work of $\mathcal{K}$ is continued from the point at which it had been abandoned. Sooner or later one of the formulae (A_n) or (B_n) is reached; this follows from our hypotheses about α and $\mathfrak{A}$, and the known nature of $\mathcal{K}$. Hence the *n*-th section will eventually be finished. $\mathcal{K}_\alpha$ is circle-free; α is computable.

□□□□□$\mathcal{K}_\alpha$□□□□$\mathcal{K}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□A_n□B_n□□□□□□□□□A_n□□□□□□□□1□□□□n□□□□□□□□□□□□□B_n□□□□□□0□□□□□□□□□□□□□□□A_n□B_n□□□□□□□$\mathcal{K}$□□□□□□□□□□□□□□□□□□A_n□B_n□□□□□□□□□□□□α□$\mathfrak{A}$□□□□□□□□$\mathcal{K}$□□□□□□□□□□□□□□n□□□□□□□□□□$\mathcal{K}_\alpha$□□□□□□α□□□□□□

□□□□□□$\mathcal{K}_\alpha$□□□

It can also be shown that the numbers α definable in this way by the use of axioms include all the computable numbers. This is done by describing computing machines in terms of the function calculus.

[illegible]

[illegible]

It must be remembered that we have attached rather a special meaning to the phrase "$\mathfrak{A}$ defines α". The computable numbers do not include all (in the ordinary sense) definable numbers. Let δ be a sequence whose n-th figure is 1 or 0 according as n is or is not satisfactory. It is an immediate consequence of the theorem of § 8 that δ is not computable. It is (so far as we know at present) possible that any assigned number of figures of δ can be calculated, but not by a uniform process. When sufficiently many figures of δ have been calculated, an essentially new method is necessary in order to obtain more figures.

[illegible]

[illegible]

[illegible]

III. This may be regarded as a modification of I or as a corollary of II.

We suppose, as in I, that the computation is carried out on a tape; but we avoid introducing the "state of mind" by considering a more physical and definite counterpart of it. It is always possible for the computer to break off from his work, to go away and forget all about it, and later to come back and go on with it. If he does this he must leave a note of instructions (written in some standard form) explaining how the work is to be continued. This note is the counterpart of the "state of mind". We will suppose that the computer works in such a desultory manner that he never does more than one step at a sitting. The note of instructions must enable him to carry out one step and write the next note. Thus the state of progress of

the computation at any stage is completely determined by the note of

[254]

instructions and the symbols on the tape. That is, the state of the system may be described by a single expression (sequence of symbols), consisting of the symbols on the tape followed by Δ (which we suppose not to appear elsewhere) and then by the note of instructions. This expression may be called the “state formula”. We know that the state formula at any given stage is determined by the state formula before the last step was made, and we assume that the relation of these two formulae is expressible in the functional calculus. In other words, we assume that there is an axiom $\mathfrak{A}$ which expresses the rules governing the behaviour of the computer, in terms of the relation of the state formula at any stage to the state formula at the preceding stage. If this is so, we can construct a machine to write down the successive state formulae, and hence to compute the required number.

III. [illegible]

[illegible]

[1] □□□“Toward Mechanical Mathematics”□*IBM Journal of Research and Development*□Vol. 4□No. 1□Jan. 1960□□2-22□□□http://www.research.ibm.com/journal/rd/04l/ibmrd0401B.pdf□□□

[2] Donald Mackenzie□“The Automation of Proof□A Historical and Sociological Exploration”□*Annals of the History of Computing*□Vol. 17□No. 3□1995□□□7-29□

[3] □□·□□□□“A Computer Program for Presburger’s Algorithm” in Jörg Seikmann and Graham Wrightson□eds., *Automation of Reasoning 1: Classical Papers on Computational Logic, 1957-1966*□Springer-Verlag□1983□□41-48□

[4] Allen Newell□J. C. Shaw□and H. A. Simon□“Empirical Explorations with the Logic Theory Machine: A Case Study in Heuristics”□*Proceedings of the Western Joint Computer Conference*□Vol. 15□1957□□218-239□□□□Edward A. Feigenbaum and Julian Feldman□eds.□*Computers and Thought*□MIT Press□1995□□109-133□

[5] □□□Michael J. Beeson□“The Mechanization of Mathematics”□in Christof Teuscher□ed.□*Alan Turing: Life and Legacy of a Great Thinker*□Springer□2004□□93□

[6] http://www.randomhouse.com/modernlibrary/100bestnonfiction.html□

[7] William Aspray□The Princeton Mathematics Community in the 1930s: An Oral-History Project□1984□4□26□□□□□□□□□J. Barkley Rosser□Stephen C. Kleene□□□□http://www.princeton.edu/~mudd/finding_aids/mathoral/pmc23.htm□

[8] □□□□□*Code: The Hidden Language of Computer Hardware and Software* □Microsoft Press□1999□□□12□□

[9] □□□·□□□□*Symbolic Logic: Part I. Elementary*□Macmillan□1896□□118□□□□□“□□□□□□□□□□□□□□□□”□□□□□

[10] □□□·□□□□“What the Tortoise Said to Achilles”□*Mind*□New Series□Vol. 4□No. 14□Apr 1895□□278-280□□□□□□□□□

[11] □□·□□□□*Science and Method*□□□□□□□Francis Maitland□Thomas Nelson & Sons□1914□Dover□2003□□147□

[12] Bew□□□□□beweisbar□□□□□□“□□□□”□——□□□

[13] □□□□□□□□□□□□□□□Kurt Gödel□*Collected Works: Volume I, Publications 1929-1936*□Oxford University Press□1986□□□□

[14] □□·□□□□“The Early History of Automated Deduction”. in Alan Robinson and Andrei Voronkov, eds.□*Handbook of Automated Reasoning*□MIT Press□2001□□Vol. 1□3-15□

[15] □□ Jean van Heijenoort□ed.□*From Frege to Gödel□A Source Book in Mathematical Logic, 1879-1931*□Harvard University Press□1967□□83-97□□□□□□*Selected Works of Giuseppe Peano*□□Hubert C. Kennedy□□□□□George Allen & Unwin□1973□□101-134□□□

[16] □□□William Ewald□ed.□*From Kant to Hilbert□A Source Book in the Foundations of Mathematics*□Oxford University Press□1996□□Vol. II□787-833□

[17] □□□□□□□□□*Grundlagen der Mathematik*□Volume I□Springer□1934□□218□

[18] □□□□□“Foundations of Mathematics”□*From Frege to Gödel*□467□

[19] □□□□□□□□□*Grundlagen der Mathematik*□Volume I□209□□□□□□□□□□□□□□□□□□□□□□

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

[illegible]1985[illegible]IEEE[illegible]IEEE[illegible]64[illegible]1[illegible]11[illegible]52[illegible]16[illegible][1][illegible]123.456[illegible]8 687 443 681 197 687[illegible]46[illegible]8 687 443 681 197 687[illegible]2^{46}[illegible]123.456[illegible]

[illegible]

[illegible]1[illegible]1/6[illegible]

[illegible]0[illegible]0[illegible]

[illegible]

[illegible]“[illegible]”[illegible]

[illegible]“[illegible]”[illegible]37.85°[illegible]

[illegible]

[illegible]0°[illegible]1°[illegible]2°[illegible]“[illegible]”[illegible]

[illegible]8[illegible]166[illegible]167[illegible]𝒟[illegible]𝒰[illegible]ℋ[illegible]“[illegible]”[illegible]

[illegible]

[illegible]180°[illegible]π[illegible]10°[illegible]π/18[illegible]——[illegible]

[illegible]π/18[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]·[illegible]·[illegible]8[illegible]7[illegible]0[illegible]

[illegible]1[illegible]

[illegible]

[illegible]10[illegible]

[illegible]10[illegible]9[illegible]10[illegible]“[illegible]”[illegible]

[illegible]10[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

10. *Examples of large classes of numbers which are computable.*

It will be useful to begin with definitions of a computable function of an integral variable and of a computable variable, etc. There are many equivalent ways of defining a computable function of an integral variable. The simplest is, possibly, as follows. If γ is a computable sequence in which 0 appears infinitely† often, and n is an integer, then let us define $\xi(\gamma,n)$ to be the number of figures 1 between the n-th and the $(n+1)$-th figure 0 in γ. Then $\phi(n)$ is computable if, for all n and come γ, $\phi(n) = \xi(\gamma,n)$.

† If $\mathcal{M}$ computes γ, then the problem whether $\mathcal{M}$ prints 0 infinitely often is of the same character as the problem whether $\mathcal{M}$ is circle-free.

10. [illegible]

[illegible]γ[illegible]0[illegible]†[illegible]n[illegible]$\xi(\gamma, n)$[illegible]γ[illegible]n[illegible]0[illegible]$(n+1)$[illegible]0[illegible]1[illegible]n[illegible]γ[illegible]$\phi(n) = \xi(\gamma, n)$[illegible]$\phi(n)$[illegible]

† [illegible]$\mathcal{M}$[illegible]γ[illegible]$\mathcal{M}$[illegible]0[illegible]$\mathcal{M}$[illegible]

[illegible]ϕ[illegible]phi[illegible]

$$\phi(n) = 2n + 1$$[illegible]

[illegible]n[illegible]

$$\phi(0) = 1$$
$$\phi(1) = 3$$

$ø(2) = 5$

$ø(3) = 7$

…

□□□□□□□□□γ□□□□□gamma□□□

0101110111101111110…

□□□□□□□□0□□□□□□□1□□□1□□□1□□□1□□1□□□□□□□□□□□□□n□□□$ø(n)$□□□□□□

γ□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$ø(n)$□□□□□□□□□□□γ□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□γ□□□□□□□□0□□□ $\xi(\gamma, n)$□□□□□xi□□□□□□□□0□□□□1□□□□□□ $ø(n)=\xi(\gamma, n)$□

□□□□□□□□□□□□0□□

□□□□□□□□□□□□$ø$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□9□□□□□□□□□□□□□□

An equivalent definition is this. Let $H(x, y)$ mean $ø(x)=y$.

□□□□□□□□□□□□□□□□□$H(x, y)$□□□$ø(x) = y$□

□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$\phi(x) = 2x+1$$

□□□□□□□□□□□I□□□$H(x, y)$□

$$(\exists z)(\text{Product}(u'', x, z) \,\&\, \text{Sum}(z, u', y))$$

□□□□□□z□□□2□□x□□z□□z□1□□y□□□□□□□□□□□

Then, if we can find a contradiction-free axiom $\mathfrak{A}_\phi$, such that $\mathfrak{A}\varphi \to P$,

□□□□□□□□□□□□□□□□□□$\mathfrak{A}_ø$□□□$\mathfrak{A}_ø \to P$□

□$\mathfrak{A}ø$□□□□□H□□□□□□□□□□□□□□Product□Sum□□P□□□□□□□□□□□□□□□□□□□

and if for each integer n there exists an integer N, such that

$$\mathfrak{A}_\phi \,\&\, F^{(N)} \to H(u^{(n)}, u^{(\phi(n))}),$$

□□□□□□□□□n□□□□□□□N□□□

$$\mathfrak{A}_\phi \,\&\, F^{(N)} \to H(u^{(n)}, u^{(\phi(n))}),$$

□□□□□$F^{(n)}$□□□□□□□□□□□□□□□N□□□□□□n□$ø(n)$□□□□□□□□□□□□□□□□n = 10□□$ø(n)$=21□□□H□□□□□1□2□□□z□□□20□□□N□□□□□□21□□□□□□□□□□□□□□□□□□□□□□□□n□$ø(n)$□□□□□

and such that, if $m \neq \phi(n)$, then, for some N',

$$\mathfrak{A}_\phi \ \& \ F^{(N')} \rightarrow \left(-H(u^{(n)}, u^{(m)}\right),$$

[illegible]$m \neq \phi(n)$[illegible]N'[illegible]

$$\mathfrak{A}_\phi \ \& \ F^{(N')} \rightarrow \left(-H(u^{(n)}, u^{(m)}\right),$$

[illegible]n[illegible]$\phi(n)$[illegible]H[illegible]N'[illegible]m[illegible]n[illegible]m[illegible]

then ϕ may be said to be a computable function.

[illegible]ϕ[illegible]

[illegible]$\mathcal{K}$[illegible]9[illegible]n[illegible]G[illegible]n[illegible]0[illegible]1[illegible]

[illegible]$\mathcal{K}_\gamma$[illegible]γ[illegible]

010111011111011111110…

[illegible]1[illegible]“1”[illegible]n[illegible]$\phi(n)$[illegible]n[illegible]n[illegible]m[illegible]

[illegible]n[illegible]m[illegible]A[illegible]B[illegible]

$$\mathbf{A}\ \mathfrak{A}_\phi \ \& \ F^{(N)} \rightarrow H(u^{(n)}, u^{(m)}) \quad \mathbf{B}\ \mathfrak{A}_\phi \ \& \ F^{(N')} \rightarrow \left(-H(u^{(n)}, u^{(m)})\right)$$

[illegible]

[illegible]n[illegible]0[illegible]n[illegible]$\phi(n)$[illegible]$H(n, m)$[illegible]n[illegible]0[illegible]m[illegible]0[illegible]m[illegible]$\phi(n)$[illegible]$H(n, m)$[illegible]

[illegible]m[illegible]B[illegible]1[illegible]m[illegible]$\phi(n)$[illegible]m[illegible]A[illegible]B[illegible]A[illegible]m[illegible]$\phi(n)$[illegible]n[illegible]n[illegible]0[illegible]m[illegible]0[illegible]

[illegible]γ[illegible]1[illegible]1[illegible]

We cannot define general computable functions of a real variable, since there is no general method of describing a real number,

[illegible]

[illegible]

but we can define a computable function of a computable variable.

[illegible]

[illegible]

[illegible]

If n is satisfactory, let γ_n be the number computed by $\mathcal{M}(n)$, and let

$$\alpha_n = \tan\left(\pi\left(\gamma_n - \tfrac{1}{2}\right)\right),$$

[255]

unless $\gamma_n = 0$ or $\gamma_n = 1$, in either of which cases $\alpha_n = 0$.

[illegible]n[illegible]γ_n[illegible]$\mathcal{M}(n)$[illegible]

$$\alpha_n = \tan\left(\pi\left(\gamma_n - 1/2\right)\right)$$

[illegible]γ_n[illegible]0[illegible]γ_n[illegible]1[illegible]α_n[illegible]0[illegible]

[illegible]0[illegible]1[illegible]γ_n[illegible]0[illegible]1[illegible]1/2π[illegible]1/2π[illegible]-90°[illegible]90°[illegible]

Then, as n runs through the satisfactory numbers, α_n runs through the computable numbers†.

† A function α_n may be defined in many other ways so as to run through the computable numbers.

[illegible]n[illegible]α_n[illegible]†[illegible]

† [illegible]α_n[illegible]

-90°[illegible]90°[illegible]–∞, ∞[illegible]-90°[illegible]90°[illegible]

[illegible]

$$\tan(x) = x + \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \frac{x^7}{3 \cdot 5 \cdot 7} + \cdots$$

x[illegible]1/2π[illegible]1/2π[illegible]

Now let $\phi(n)$ be a computable function which can be shown to be such that for any satisfactory argument its value is satisfactory‡. Then the function f, defined by $f(\alpha_n)=\alpha_{\phi(n)}$, is a computable function and all computable functions of a computable variable are expressible in this form.

‡ Although it is not possible to find a general process for determining whether a given number is satisfactory, it is often

possible to show that certain classes of numbers are satisfactory.

[illegible]$\phi(n)$[illegible][†][illegible]$f(\alpha_n)=\alpha_{\phi(n)}$[illegible]$f$[illegible]

[†] [illegible]

[illegible]$\phi(n)$[illegible]$\phi(n)$[illegible]$\phi(n)$[illegible]x[illegible]x[illegible]1[illegible]$2x + 1$[illegible]

Similar definitions may be given of computable functions of several variables, computable-valued functions of an integral variable, etc.

I shall enunciate a number of theorems about computability, but I shall prove only (ii) and a theorem similar to (iii).

[illegible]

[illegible](ii)[illegible](iii)[illegible]

[illegible]10[illegible](ii)[illegible](iii)[illegible]

(i) A computable function of a computable function of an integral or computable variable is computable.

(i) [illegible]

[illegible]

(ii) Any function of an integral variable defined recursively in terms of computable functions is cpmputable. *I.e.* if $\phi(m,n)$ is computable, and r is some integer, then $\eta(n)$ is computable, where

$$\eta(0) = r,$$

$$\eta(n) = \phi\left(n, \eta(n-1)\right).$$

(ii)[illegible]$\phi(m, n)$[illegible]r[illegible]$\eta(n)$[illegible]

$$\eta(0) = r,$$

$$\eta(n) = \phi(n, \eta(n-1))$$

□□□□□□□*η*□□□eta□□□□□*n*□□□□□“□□□□□□□□□□”□*η*□“□□□□□□□”□□□ø(*m*, *n*)□□□□□□□□□□□*r* = 1□□□ø(*m*, *n*)□□□□□□

$$\phi(m, n) = m \cdot n$$

□□□ø(*m*, *n*)□□□□□□□□□□□*η*□□□□□

$\eta(0) = r = 1$

$\eta(1) = ø(1, \eta(0)) = ø(1, 1) = 1·1 = 1$

$\eta(2) = ø(2, \eta(1)) = ø(2, 1) = 2·1 = 2$

$\eta(3) = ø(3, \eta(2)) = ø(3, 2) = 3·2 = 6$

$\eta(4) = ø(4, \eta(3)) = ø(4, 6) = 4·6 = 24$

$\eta(5) = ø(5, \eta(4)) = ø(5, 24) = 5·24 = 120$

…

□□*η*(*n*)□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□ø(*m*, *n*)□□□□□□□□□□*η*(*n*□□□□□□□□□□□

(iii) If ø(*m*, *n*) is a computable function of two integral variables, then ø(*n*, *n*) is a computable function of *n*.

(iii) □□ ø(*m*, *n*)□□□□□□□□□□□□□□□□□□□□ø(*n*, *n*)□□□□□□□□□□□

□□

(iv) If ø(*n*) is a computable function whose value is always 0 or 1, then the sequence whose *n*-th figure is ø(*n*) is computable.

(iv) □□ø(*n*)□□□□□□□□0□1□□□□□□□□□□□□*n*□□□□ø(*n*)□□□□□□□□□□

□□□□□□□IsPrime(*n*)□*n*□□□□□1□□□□□□□0□□□□□□□□□□□□□□□□□

00110101000101000101000100000010…

□□□□□*n*□0□□□□31□□□□□□□2□3□5□7□11□13□17□19□23□29□□□□□□□□□1□

□□□□□□□□□□□□□□□□□□□G. H. □□□□□□□□□□□□1□□□□□□□□□□1931□□

□□□□□ □□□□□□□□□□□□□□□□□□□□L□R□

(i) □□□□□□□□□□□□□□□

(ii) □□□□□□□□□□□□

(iii) L□□□□□□□□□□R□□□□□□□□

□□□□□□*α*□□□□□□□□□□□□□□L□□□□□□□□□□□□R□□□*α*□□□□□□L□R□□□□□□□[2]

□□□

□□L□□□□□□□□□□□□□R□□

Dedekind's theorem does not hold in the ordinary form if we replace “real” throughout by “computable”.

But it holds in the following form:

(v) If $G(\alpha)$ is a propositional function of the computable numbers and

$$(a)\quad (\exists\alpha)\,(\exists\beta)\Big\{G(\alpha)\ \&\ \big(-G(\beta)\big)\Big\},$$

$$(a)\quad (\exists\alpha)\,(\exists\beta)\Big\{G(\alpha)\ \&\ \big(-G(\beta)\big)\Big\},$$

$$(b)\quad G(\alpha)\ \&\ \big(-G(\beta)\big) \rightarrow (\alpha < \beta),$$

$$(b)\quad G(\alpha)\ \&\ \big(-G(\beta)\big) \rightarrow (\alpha < \beta),$$

and there is a geral process for determining the truth value of $G(\alpha)$, then [256] there is a computable number ξ such that

[256]

there is a computable number ξ such that

$$G(\alpha) \to \alpha \leq \xi,$$

$$-G(\alpha) \to \alpha \geq \xi.$$

[illegible]$G(\alpha)$[illegible] ξ [illegible]

$$G(\alpha) \to \alpha \leq \xi,$$

$$-G(\alpha) \to \alpha \geq \xi.$$

“[illegible]$G(\alpha)$[illegible]”[illegible]ξ[illegible]$G(\alpha)$[illegible]$G(\alpha)$[illegible]ξ[illegible]

In other words, the theorem holds for any section of the computables such that there is a general process for determining to which class a given number belongs.

[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

Owing to this restriction of Dedekind's theorem, we cannot say that a computable bounded increasing sequence of computable numbers has a computable limit. This may possibly be understood by considering a sequence such as

$$-1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{16}, \frac{1}{2}, \ \dots .$$

[illegible]

$$-1, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, -\frac{1}{16}, \frac{1}{2}, \ \dots .$$

[illegible]sci.logic[illegible]http://sci.techarchive.net/Archive/sci.logic/2004-08/2244.html[illegible]1/2[illegible]1/32[illegible]0[illegible]0[illegible]

[illegible]1[illegible]1[illegible]

On the other hand, (v)enables us to prove

(vi) If α and β are computable and $\alpha < \beta$ and $\varnothing(\alpha) < 0 < \varnothing(\beta)$, where $\varnothing(\alpha)$ is a computable increasing continuous

function, then there is a unique computable number γ, satisfying $\alpha < \gamma < \beta$ and $ø(\gamma) = 0$.

□□□□(v)□□□□□□□□

(vi)□□α□β□□□□□□□α<β□$ø(\alpha)$<0<$ø(\beta)$□□□□$ø(\alpha)$□□□□□□□□□□□□□□□□□□□□□□□□□γ□□α<γ<β□□$ø(\gamma)$=0□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$\phi(x) = 5x^3 - 3x^2 - 7$$

□□x□□□□□□□□□□□0□

□□□□□$ø(x)$□□□□□$ø(1)$=-5□ $ø(2)$=21□□□□□□□□□□□□□□□□□□□□□1□2□□□

□□1□2□□□□□□□□□α□β□□□□□□$\alpha < \beta$□$ø(\alpha)$<0<(β)□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□γ□□□□α□β□□□□$ø(\gamma)$□□□0□□□□□□□□□□□□γ□□□□□□α□β□□□□□□□□□□□1.5□□□$ø(1.5)$□□□□3.125□□□3.125□□0□□□□□□1.5□□□□β□□□□□□□□□□□1□1.5□□□□□□□□□□□1.25□$ø(1.25)$ = (1.921875□□□□□0□□□□1.25□□□□α□□□□□□□□□□□□□□□1.25□1.5□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□“□□□”□□□

$\alpha_0 = 3$

$\alpha_1 = 3.1$

$\alpha_2 = 3.14$

$\alpha_3 = 3,141$

$\alpha_4 = 3,1415$

$\alpha_5 = 3/14159$

…

□□a_3□a_4□□□0.0005□□a_4□a_5□□□0.00009□

□□□□□□□□□□□□□□□□□□ε□□□epsilon□□□□□□□□□□□□□□□□ε□□□0.0001□□□□□ε□□□□□□□□□□□N□□□□□□□$n > N$□$m > N$□□□□a_n□a_m□□□□□□□□□□□□□□□□□□□□□□□□□ $\left|a_n - a_m\right| < \varepsilon$ □□□□□□□□□□□□□□□□□□□□□□□ε□□□0.0001□N□3□□□ $\left|a_4 - a_5\right| < 0.0001$ □□n□m□□3□□a_n□a_m□□□□□□□□□0.0001□

□□□□□□□□□□□□□□□□□□□□□

Computable convergence.

We shall say that a sequence β_n of computable numbers *converges computably* if there is a computable integral valued function $N(\eta)$ of the computable variable η, such that we can show that, if $\eta > 0$ and $n > N(\eta)$ and $m > N(\eta)$, then $|\beta_n - \beta_m| < \epsilon.$

□□□□□

□□□□□□□□□□ε□□□□□□□□□□□□□$N(\varepsilon)$□□□□□□□ $\varepsilon>0$， $n>N(\varepsilon)$， $m>N(\varepsilon)$， 则 $\left|\beta_n-\beta_m\right|<\varepsilon$ ，□□□□□□□□□□□□□□□□□β_n□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□ε□□□□□□□□$N(\varepsilon)$□□□□□□□□

We can then show that

(vii) A power series whose coefficients form a computable sequence of computable numbers is computably convergent at all computable points in the interior of its interval of convergence.

□□□□□□□□

(vii) □□

□□□□□□□□□□□

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

□□□

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

□□□□□□□a_i□□□□1□0□-1/3!□0□1/5!□0□1/7!□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□x□□0□□□□□□□□□□□□x□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□x□□

(viii) The limit of a computably convergent sequencee is computable.

(viii) □□□□□□□□□□□□□□□□□□□□

□□□

And with the obvious definition of "uniformly computably convergent":

(ix) The limit of a uniformly computably convergent computable sequence of computable functions is a computable function. Hence

(x) The sum of a power series whose coefficients form a computable sequence is a computable function in the interior of its interval of convergence.

□□“□□□□□□□”□□□□□□□□

(ix) □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

(x) □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

From (vii) and $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \ldots)$ we deduce that π iscomputable.

From $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots$ We deduce that *e* is computable.

[257]

From (vi) we deduce that all real algebraic numbers are computable.

□(viii)□ $\pi = 4(1 - \frac{1}{3} + \frac{1}{5} - \cdots)$, , □□□□□π□□□□□□

□ $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$ □□□□e□□□□□□

□(vi)□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□

From (vi) and (x) we deduce that the real zeros of the Bessel functions are computable.

□(vi)□(x)□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□0□□□

□□□□□□□□□□□(ii)□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

Proof of (ii).

(ii)□□□□

□□10□□□□□□□□□□□□□□□□$H(x, y)$□□□$\phi(x) = y$□□$H(x, y)$□□□□□□□□□□$m \neq \phi(n)$□□□□□□□□□□□□$H(u^{(n)}, u^{(\phi(n))})$□$-H(u^{(n)}, u^{(m)})$□□□□□□□□

□□□□□$\phi(x)$□□□□□□

□□□□□□□□□□□□□□□□□□□$\eta(x)$□□□$\phi(x)$□□□□

$$\eta(0) = r$$

$$\eta(n) = \phi(n, \eta(n-1))$$

Let $H(x, y)$ mean "$\eta(x)=y$", and let $K(x, y, z)$ mean "$\phi(x, y)=z$".

□$H(x, y)$□□$\eta(x) = y$□□□□$K(x, y, z)$□□$\phi(x, y) = z$□

□□□□□□□□

$$\phi(x, y) = x \cdot y$$

□□$K(x, y, z)$□□Product(x, y, z)□

$\mathfrak{A}_\phi$ is the axiom for $\phi(x, y)$.

$\mathfrak{A}_\phi$□$\phi(x, y)$□□□□

□□$\mathfrak{A}_\varphi$□□□□□□□□$K(x, y, z)$□□□□□□□□□□□□□□□□□□□□□□□□□Sum□Product□□□□□$\eta(x)$□□□□□□□□

We that $\mathfrak{A}_\eta$ to be

$$\mathfrak{A}_\phi \;\&\; P \;\&\; \big(F(x,y) \to G(x,y)\big) \;\&\; \big(G(x,y) \;\&\; G(y,z) \to G(x,z)\big)$$
$$\&\; \big(F^{(r)} \to H(u, u^{(r)})\big) \;\&\; \big(F(v,w) \;\&\; H(v,x) \;\&\; K(w,x,z) \to H(w,z)\big)$$
$$\&\; \big[H(w,z) \;\&\; G(z,t) \vee G(t,z) \to \big(-H(w,t)\big)\big].$$

□□□$\mathfrak{A}_\eta$□□□□

$$\mathfrak{A}_\phi \;\&\; P \;\&\; \big(F(x,y) \to G(x,y)\big) \;\&\; \big(G(x,y) \;\&\; G(y,z) \to G(x,z)\big)$$
$$\&\; \big(F^{(r)} \to H(u, u^{(r)})\big) \;\&\; \big(F(v,w) \;\&\; H(v,x) \;\&\; K(w,x,z) \to H(w,z)\big)$$
$$\&\; \big[H(w,z) \;\&\; G(z,t) \vee G(t,z) \to \big(-H(w,t)\big)\big].$$

□□□□□□□□□□□G□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□"□□"□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□*Proceedings of the London Mathematical Society*□□□□□□□□□□□□□□[3]□□□□□□□□□□□□□□□□□□□□16□□□□□□□□□□□□□□□$G(x, y)$□□□□□"x□y□□"□$y > x$□□□□□□□F□□□□□□□□

$\mathfrak{A}_\eta$□7□□□□□□□□□$\mathfrak{A}_\phi$□P□□□□□□□□□□□□□□y□x□□□□□□y□□x□□□□□□□□□□□□□□G□□□□□□□□□y□□x□z□□y□□□z□□x□□

□□□□□□□□□□□□ $H(u, u^{(r)})$□□□□□□□$\eta(0) = r$□□□□□□□□□

$$(F(v,w)\ \&\ H(v,x)\ \&\ K(w,x,z) \rightarrow H(w,z))$$

□□□□□□□□□□□□□□$w = v + 1$□□$\eta(v) = x$□$\phi(w, x) = z$ □□□ $\eta(w) = z$, □□:

$$\eta(v+1) = \phi(v+1, \eta(v))$$

□□□□□□□n□□0□$\eta(n)$□□□□□□□□□

$\mathfrak{A}_\eta$□□□□□□□□□□

$$[H(w,z)\ \&\ G(z,t)\ v\ G(t,z) \rightarrow (-H(w,t))]$$

□□□□□□□G□□□□□□□□□□□□□□□G□□□□□□□□□□□□□□□z□□□t□□G□□□□□□□□□□□□□□□□□□□□□□□$H(w, z)$□□□□□□□□□□□z□t□$H(w, t)$□□□□□□□$\eta(w) \neq t$□

□□$\mathfrak{A}_\eta$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

I shall not give the proof of consistenc of $\mathfrak{A}_\eta$. Such a proof may be constructed by the methods used in Hilbert and Bernays, *Grundlagen der Mathematik* (Berlin, 1934), p. 209 *et seq*. The consistency is also clear from the meaning.

□□□□□$\mathfrak{A}_\eta$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□Berlin□1934□□□□209□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□·□□□□□□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□1933□□□□□□□□□□□□1934□□□□□□□□□□□□□□□□2□□1939□□“□□□□□□□□□□□□□□”□□□□·□□□□□□□□□□□[4]□□□□□□□□□□□□□□□□□□□□□□□□□[5]□□□□□□□□“□□□□□□·□□□□□□□□□□□□□”□

□□□□□□1935~1936□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□R□□□□□□□□G□□

$$(x) - R(x,x)$$

$$(x)(y)(z)(R(x,y)\ \&\ R(y,z) \rightarrow R(x,z))$$

$$(x)(\exists y) R(x,y)$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□0□1□□S□□□□□

$$(x)(\exists y)S(x,y)$$

$$(\exists x)(y)-S(x,y)$$

$$(x)(y)(r)(s)\big(S(x,r)\,\&\,S(y,r)\,\&\,S(s,x)\to S(s,y)\big)$$

[illegible]

[illegible]0[illegible]n[illegible]$n + 1$[illegible]n[illegible]n[illegible]$n + 1$[illegible]0[illegible]0 + 1[illegible]1[illegible]1[illegible]1+1[illegible]2[illegible]

[illegible]0[illegible]n-1[illegible]n[illegible]

Suppose that, for some n, N, we have shown

$$\mathfrak{A}_\eta \,\&\, F^{(N)} \to H(u^{(n-1)}, u^{(\eta(n-1))}),$$

[illegible]n[illegible]N[illegible]

$$\mathfrak{A}_\eta \,\&\, F^{(N)} \to H\left(u^{(n-1)}, u^{(\eta(n-1))}\right),$$

[illegible]$\mathfrak{A}_\phi$[illegible]K[illegible]

then, for some M,

$$\mathfrak{A}_\phi \,\&\, F^{(M)} \to K(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}),$$

[illegible]M[illegible]

$$\mathfrak{A}_\phi \,\&\, F^{(M)} \to K\left(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}\right),$$

[illegible]$\mathfrak{A}_\eta \to \mathfrak{A}_\phi$[illegible]

$$\mathfrak{A}_\eta \,\&\, F^{(M)} \to K\left(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}\right)$$

[illegible]$F^{(M)}$[illegible]H[illegible]

$$\mathfrak{A}_\eta \ \& \ F^{(M)} \to F(u^{(n-1)}, u^{(n)}) \ \& \ H\big(u^{(n-1)}, u^{(\eta(n-1))}\big) \ \& \ K\big(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}\big),$$

$$\mathfrak{A}_\eta \ \& \ F^{(M)} \to F\left(u^{(n-1)}, u^{(n)}\right) \& \ H\left(u^{(n-1)}, u^{(\eta(n-1))}\right) \& K\left(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}\right),$$

□□□□□□$\mathfrak{A}_\eta$□□□□□□□□□□□□□□□□□

and

$$\mathfrak{A}_\eta \ \& \ F^{(M)} \to [F(u^{(n-1)}, u^{(n)}) \ \& \ H(u^{(n-1)}, u^{(\eta(n-1))}) \ \& \ K(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}) \to H\big(u^{(n)}, u^{(\eta(n))}\big)].$$

□□□

$$\mathfrak{A}_\eta \ \& \ F^{(M)} \to \Big[F\left(u^{(n-1)}, u^{(n)}\right) \& \ H\left(u^{(n-1)}, u^{(\eta(n-1))}\right) \& K\left(u^{(n)}, u^{(\eta(n-1))}, u^{(\eta(n))}\right) \to H\left(u^{(n)}, u^{(\eta(n))}\right)\Big].$$

□□□□□□□□□□□$\mathfrak{A}_\eta$□□□□□□□□□□□□□□ $u^{(n)}$□ $u^{(n-1)}$□$u^{(\eta(n))}$□$u^{(\eta(n-1))}$□□□□□□w□v□z□x□□□□□□□□□

Hence $\mathfrak{A}_\eta \ \& \ F^{(M)} \to H(u^{(n)}, u^{(\eta(n))})$.

□□□ $\mathfrak{A}_\eta \ \& \ F^{(M)} \to H\left(u^{(n)}, u^{(\eta(n))}\right)$。 □

□□□□□□□□□□□□□□□□□□□□□□□0□□□□□□□□□□□□□

Also $\mathfrak{A}_\eta \ \& \ F^{(r)} \to H(u, u^{(\eta(0))})$.

□□□ $\mathfrak{A}_\eta \ \& \ F^{(r)} \to H\left(u, u^{(\eta(0))}\right)$。□

□□□□□□□□□$\eta(0)$□□□□r□□□□□□□□□□□□n□□□□□

Hence for each n some formula of the form

$$\mathfrak{A}_\eta \ \& \ F^{(M)} \to H(u^{(n)}, u^{(\eta(n))})$$

is provable.

□□□□□□□□n□□□□□□□□□□

$$\mathfrak{A}_\eta \,\&\, F^{(M)} \rightarrow H\left(u^{(n)}, u^{(\eta(n))}\right)$$

□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□$\eta(u)$□□□□□□□H□□□□

Also, if $M' \geq m$ and $M' \geq m$ and $m \geq \eta(u)$, then

$$\mathfrak{A}_\eta \,\&\, F^{(M')} \rightarrow G(u^{\eta((n))}, u^{(m)}) \text{ v } G(u^{(m)}, u^{(\eta(n))})$$

□□□□□ M'□□□

$$\mathfrak{A}_\eta \,\&\, F^{(M')} \rightarrow G\left(u^{(\eta(n))}, u^{(m)}\right) \text{v } G\left(u^{(m)}, u^{(\eta(n))}\right)$$

□□□□□□□ $m \neq \eta(n)$□□□m□□□□□□□□$\eta(n)$□

[258]

and

$$\mathfrak{A}_\eta \,\&\, F^{(M')} \rightarrow \left[\{G(u^{(\eta(n))}, u^{(m)}) \text{ v } G(u^{(m)}, u^{(\eta(n))}) \\ \& H(u^{(n)}, u^{(\eta(n))}\} \rightarrow \left(-H(u^{(n)}, u^{(m)})\right)\right].$$

□□

$$\mathfrak{A}_\eta \,\&\, F^{(M')} \rightarrow \left[\left\{G\left(u^{(\eta(n))}, u^{(m)}\right) \text{v } G\left(u^{(m)}, u^{(\eta(n))}\right) \\ \& H\left(u^{(n)}, u^{(\eta(n))}\right)\right\} \rightarrow \left(-H\left(u^{(n)}, u^{(m)}\right)\right)\right].$$

□□□□□□□□□□□$\mathfrak{A}_\eta$□□□□□□□□□□□□□□□□□□□□□□□□□$u^{(n)}$, $u^{(m)}$□$u^{(\eta(n))}$
□□□□□w□t□z□□□□□□□□□□

Hence $\mathfrak{A}_\eta \,\&\, F^{(M')} \rightarrow \left(-H\left(u^{(n)}, u^{(m)}\right)\right).$

The conditions of our second definition of a computable function are therefore satisfied.

□□ $\mathfrak{A}_\eta \,\&\, F^{(M')} \rightarrow \left(-H(u^{(n)}, u^{(m)})\right).$

□□□□□□□□□□□□□□□□□□□□□

□□□“□□□□□”□□□□220□□“□□□□□□□□□□□”□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□

$$\mathfrak{A}_\eta \ \& \ F^{(M)} \rightarrow H\left(u^{(n)}, u^{(\eta(n))}\right)$$

$$\mathfrak{A}_\eta \ \& \ F^{(M')} \rightarrow \left(-H\left(u^{(n)}, u^{(m)}\right)\right)$$

□□□□□n□m□□□□□□□□□□□

Consequently η is a computable function.

□□□η□□□□□□□□□

□□□(iii)□□□□□□□□□□$\phi(m, n)$□□□□□□□□□□□□□□□□□□□□$\phi(n, n)$□n□□□□□□□

□□n□□0□1□2□□□□□□□□□□□□□□□□□n□□□□□□□□□□□□□□□□□□

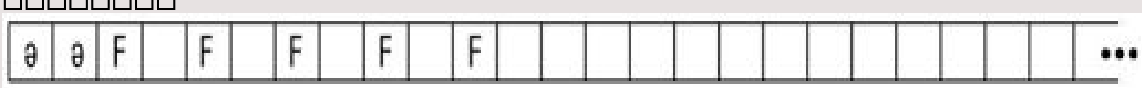
Proof of a modified form of (iii).

Suppose that we are given a machine $\mathcal{N}$, which, starting with a tape bearing on it əə followed by a sequence of any number of letters "*F*" on *F*-squares and in the *m*-configuration *b*, will compute a sequence γn depending on the number *n* of letters "*F*".

□□(iii)□□□□□

□□□□□□□□$\mathcal{N}$□□□□□□□□m-□□□b□□□□□əə□□□□□□F-□□□□□□□"F"□□□□□□□"F"□□□n□□□□γ_n□

□□□□□□□□



□□□"F"□□□γ_m□□□□□——□□n□□□□□□□□□□□□5□"F"□□□□□□□□□□□□□□□□□□□□□□□□□□□5□□□□

If $\phi n(n)$ is the *m*-th figure of γn then the sequence β whose *n*-th figure is $\phi n(n)$ is computable.

□□$\phi_n(m)$□γ_n□m□□□□□□□□n□□□□ϕ_n□β□□□□□□□□□

□□□□□ϕ_5□□5□□□□□□12□□□□□□□□□□□□□□□□□□□□□□β□□□□□n□□□ϕ_n□□□□□□□□□□□□□□□β□□□□□□□□□1□□□□□□□□□□□□□□□□□□□2□□□□□□□□□□□□□□□□

□□

□□□□□□□□□□□□□□F□□□□□□□□□□□□□□□□□□□F□□□F□□□□□□□□□□□n□□n□□□F□□□□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□F□□□□□□□□□□□□□□□

□□□F-□□□□□□0□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

We suppose that the table fir $\mathcal{N}$ has been written out in such a way that in each line only one operation appears in the operations column.

□□□□$\mathcal{N}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□Ξ□□□xi□□Θ□□□theta□□□□□□□□□□□□□□□□□□□□□□□

We also suppose that Ξ, Θ, $\overline{0}$, and $\overline{1}$ do not occur in the table, and we replace ə throughout by Θ, 0 by $\overline{0}$, and 1 by $\overline{1}$.

□□□□Ξ□Θ□$\overline{0}$□$\overline{1}$□□□□□□□□□□□□□ə□□□Θ□0□□□$\overline{0}$□1□□□$\overline{1}$□

□□□□"□ə□□□Θ"□□□ə□□□□□□Θ□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□*h*□*k*□□□□

$\mathfrak{c}$、$\mathfrak{u}$、$\mathfrak{u}_1$、$\mathfrak{u}_2$、$\mathfrak{u}_3$、$\mathfrak{v}$、$\mathfrak{v}_1$、$\mathfrak{v}_2$、$\mathfrak{v}_3$□□□□□□□□□□□□

Further substitutions are then made. Any line of form

$$\mathfrak{A} \qquad \alpha \qquad P\overline{0} \qquad \mathfrak{B}$$

we replace by

$$\mathfrak{A} \qquad \alpha \qquad P\overline{0} \qquad \mathfrak{re}(\mathfrak{B}, \mathfrak{u}, h, k)$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$\mathfrak{A} \qquad \alpha \qquad P\overline{0} \qquad \mathfrak{B}$$

□□□□□□□□

$$\mathfrak{A} \qquad \alpha \qquad P\overline{0} \qquad \mathfrak{re}(\mathfrak{B}, \mathfrak{u}, h, k)$$

□□□□□□□□□□□0□□□□$\overline{0}$□□□$\mathfrak{A}$□$\mathfrak{B}$□□□□□□*α*□□□□□□□□□□□□□□□□□□□□$\mathfrak{re}$□□□"□□"□□□□□$\mathfrak{re}$□*k*□□□□□□□□□□□□*h*□□□□□□□□$\mathfrak{B}$□□□□□□□□□□□□*h*□□□□□□□□$\mathfrak{u}$□

□□□4□□□□□$\mathfrak{re}$□□□□□□□□□□□□□□□□□□□$\mathfrak{f}$□□□□□□□□□□□□□□□□□□□□□Θ□□□□□□□

□□□F□□□□□□□□□□□□□*h*□□□□□□□F□□□□□□□□□□□□□□□□□□□□□□*u*□□□□*h*□□$\overline{0}$□□□□0□

and any line of the form

$$\mathfrak{A} \qquad \alpha \qquad P\overline{1} \qquad \mathfrak{B}$$

by $\mathfrak{A} \qquad \alpha \qquad P\overline{1} \qquad \mathfrak{re}(\mathfrak{B}, \mathfrak{v}, h, k)$

□□□□□□□□□□□□□

$$\mathfrak{A} \qquad \alpha \qquad P\overline{1} \qquad \mathfrak{B}$$

□□□

$\mathfrak{A}$	α	$P\overline{1}$	$\mathfrak{re}(\mathfrak{B}, \mathfrak{v}, h, k)$

□□u□$\mathfrak{v}$□□□□□□□□□□□□□□0□1□□□□□□□□□□□F□□□□□□□□□□□□□□□□□□□□□□□□$\mathfrak{u}$□$\mathfrak{v}$□□□□□□□□□□

and we add to the table the following lines:

$\mathfrak{u}$		$\mathfrak{pe}(\mathfrak{u}_1, 0)$
$\mathfrak{u}_1$	$R, Pk, R, P\Theta, R, P\Theta$	$\mathfrak{u}_2$
$\mathfrak{u}_2$		$\mathfrak{re}(\mathfrak{u}_3, \mathfrak{u}_3, k, h)$
$\mathfrak{u}_3$		$\mathfrak{pe}(\mathfrak{u}_2, F)$

□□□□□□□□□□□□□□□□□

$\mathfrak{u}$		$\mathfrak{pe}(\mathfrak{u}_1, 0)$
$\mathfrak{u}_1$	$R, Pk, R, P\Theta, R, P\Theta$	$\mathfrak{u}_2$
$\mathfrak{u}_2$		$\mathfrak{re}(\mathfrak{u}_3, \mathfrak{u}_3, k, h)$
$\mathfrak{u}_3$		$\mathfrak{pe}(\mathfrak{u}_2, F)$

□□□$\mathfrak{u}$□□□□□□□□□□0□□$\mathfrak{u}_1$□□□□k□□□Θ□□□□□□□□□□□□□□□□□□□□□□□$\mathfrak{u}_2$□$\mathfrak{u}_3$□h□□□□k□□□□□□□□□□□□□h□□□□□□k□□□□□□□□□□□h□k□□□□□□□□□□□□□□□F□□□□□□□□□□□□□□□□□□□□□□□□□$\mathfrak{u}_2$□□□□□$\mathfrak{u}_3$□□$\mathfrak{u}_3$□□□□□□$\mathfrak{u}_2$□□□□□□□□□□□□

$$\mathfrak{re}(\mathfrak{u}_3, \mathfrak{b}, k, h)$$

□□□□□□□□□□□□h□□□k□□□□□□□□□□□$\mathfrak{b}$□□□□□□□□$\mathfrak{b}$□□□□□□□□□□□□□□□$\mathfrak{b}$□□□□□□□□□

ə	ə	Ξ	Θ	Θ	…			0	h	Θ	Θ	F	…			0	h	Θ	Θ	F	F			•••

□

□□□1□□□0□

and similar lines with $\mathfrak{v}$ for u and 1 for 0 together with the following line

$\mathfrak{c}$	$R, P\Xi, R, Ph$	$\mathfrak{b}$.

□□□$\mathfrak{v}$□□u□1□□0□□□□□□□□□□□

$\mathfrak{c}$	$R, P\Xi, R, Ph$	$\mathfrak{b}$.

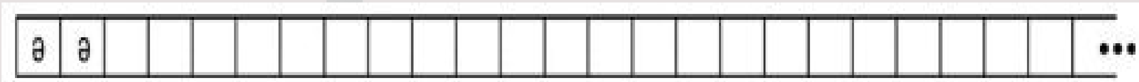
□□□□□□□□□□□□□□□□□□□□□□□□□□h□□□□□□□□□□□□□□□

$$\mathfrak{c} \qquad R, P\Xi, R, P\Theta, R, P\Theta$$

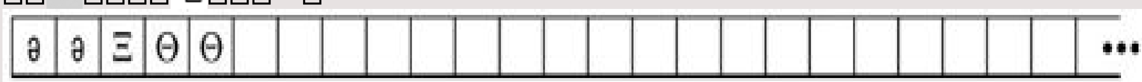
We then have the table for the machine $\mathcal{N}'$ which computes β. The initial *m*-configuration is $\mathfrak{c}$, and the initial scanned symbol is the second ə.

[illegible]β[illegible]$\mathcal{N}'$[illegible]m-[illegible]$\mathfrak{c}$[illegible]ə[illegible]

[illegible]$\mathcal{N}'$[illegible]$\mathcal{N}$[illegible]$\mathcal{N}'$[illegible]F[illegible]$\mathcal{N}'$[illegible]$\mathfrak{c}$[illegible]



[illegible]$\mathfrak{c}$[illegible]Ξ[illegible]Θ[illegible]



Ξ[illegible]$\mathfrak{b}$[illegible]Θ[illegible]F[illegible]0[illegible]$\bar{0}$[illegible]*h*[illegible]*k*[illegible]*h*[illegible]$\mathfrak{u}$[illegible]0[illegible]*k*[illegible]Θ[illegible]



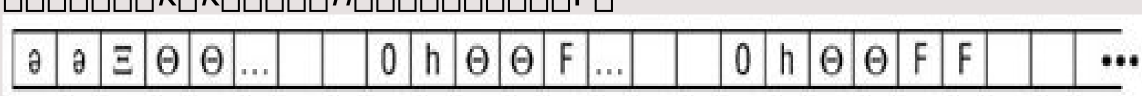
[illegible]$\mathfrak{u}_2$[illegible]$\mathfrak{u}_3$[illegible]*k*[illegible]*h*[illegible]F[illegible]



[illegible]*h*[illegible]*k*[illegible]$\mathfrak{b}$[illegible]1[illegible]1[illegible]$\bar{1}$[illegible]*h*[illegible]*k*[illegible]0[illegible]$\bar{0}$[illegible]*h*[illegible]$\mathfrak{u}$[illegible]0[illegible]*k*[illegible]Θ[illegible]



[illegible]*k*[illegible]*k*[illegible]*h*[illegible]F[illegible]



[illegible]F[illegible]F-[illegible]*h*[illegible]*k*[illegible]$\mathfrak{b}$[illegible]2[illegible]

[illegible]10[illegible]

[1] □□□□□*Code□The Hidden Language of Computer Hardware and Software*□Microsoft Press□1999□□□23□□

[2] G. H. □□□□□□□□□□□□*A Course of Pure Mathematics*□10th edition□Cambridge University Press□1952□□□□□□□□□□□□□□□□□30□□□□□□□□□□□□□□□□1□□1908□□□□□□□□6□□1933□□□□□□□□□□□□□□□□□□□□□1931□□□□□□□□□□□□□□□□□□□□□□□□□□□□5□□1928□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□*The Principles of Mathematics*□Cambridge University Press□1903□□□34□□*Introduction to Mathematical Philosophy*□George Allen & Unwin□1919□1920□□□7□□E. W. Hobson□□*The Theory of Functions of a Real Variable and the Theory of Fourier's Series*□□1□□□□□□□□□□8□□□□□□□□□□□□□□□□1872□□□□□□□□□□□□"*Stetigkeit und Irrationale Zahlen*"□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□"on the Theory of Numbers"□□□□Wooster W. Beman□Open Court Press□1901□Dover□1963□□□□□□□□□□□"Continuity and Irrational Numbers"□□□□□□□□William Ewald□*From Kant to Hilbert: A Source Book in the Foundations of Mathematics*□Oxford University Press□□□□□□□□□□□□□□Vol. II□765-779□

[3] □□·□□□"On Computable Numbers, with an Application to the Entscheidungsproblem. A Correction"□Proceedings of the London Mathematical Society□2nd Series□Vol. 43□1937□□544-546□

[4] □□□·□□□"A Note on the Entscheidungsproblem"□The Journal of Symbolic Logic□Vol. 1□No. 1□ Mar. 1936□□40-41□

[5] □□□·□□□"Correction to A Note on the Entscheidungsproblem"□The Journal of Symbolic Logic□Vol. 1□No. 3□Sept. 1936□□101-102□

[illegible]14[illegible]
[illegible]

[illegible]

[illegible]

[illegible]“[illegible]”[illegible]

[illegible]8[illegible]0[illegible]9[illegible]10[illegible]

[illegible]11[illegible]0[illegible]0[illegible]

[259]

11. *Application to the Entscheidungsproblem.*

The results of §8 have some important applications. In particular, they can be used to show that the Hilbert Entscheidungsproblem can have no solution. For the present I shall confine myself to proving this particular theorem. For the formulation of this problm I must refer the reader to Hilbert and Ackermann's *Grundzüge der Theoretischen Logik* (Berlin, 1931), chapter 3.

11. [illegible]

§8[illegible]*Grundzüge der Theoretischen Logik*[illegible]Berlin[illegible]1931[illegible]3[illegible]

[illegible]1928[illegible]120[illegible]3[illegible]Der engere Funktionenkalkül[illegible]“[illegible]”[illegible]

> *Das Entscheidungsproblem ist gelöst, wenn man ein Verfahren kennt, das bei einem vorgelegten logischen Ausdruck durch endlich viele Operationen die Entscheidung über die Allgemeingültigkeit bzw. Erfüllbarkeit erlaubt.… [Das] Entscheidungsproblem mu ß als das Hauptproblem der mathematischen Logik bezeichnet werden.*[1]
>
> □□□……□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□validity□□□□□□□satisfiability□□□□□□□□□□□□□□□□25□□□□□·□□□□□□□□□□□□□□□□□□□□□□□North-Holland□□□□□1954□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□**K**□□□□□Kalkül□□□□□□□□□□□□□□□□□□□□

> I propose, therefore, to show that there can be no general process for determining whether a given formula $\mathfrak{A}$ of the functional calculus **K** is provable, *i.e.* that there can be no machine which, supplied with any one $\mathfrak{A}$ of these formulae, will eventually say whether $\mathfrak{A}$ is provable.

> □□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□**K**□□□$\mathfrak{A}$□□$\mathfrak{A}$□□□□□□□□$\mathfrak{A}$□□□□□□

□□□□□"□□□"□□□□"□□□"□"□□□□"□□1930□□□□□□□□□□□□□

□□□9□□□10□□□

□1936□□□unentscheidbare Sätze□"□□□□□□□"□□□"□□□□□□□□□□□□□□□□□□□□□□□□□"□□□□57□□□□□□□□□□□□□□□□□□□□□□□□

> It should perhaps be remarked that what I shall prove is quite different from the well-known results of Gö†. Gödel has shown that (in the formalism of Principia Mathematica) there are propositions $\mathfrak{A}$ such that neither $\mathfrak{A}$ nor — $\mathfrak{A}$ is provable. As a consequence of this, it is shown that no proof of consistency of Principia Mathematica (or of **K**) can be given within that formalism. On the other hand, I shall show that there is no general method which tells whether a given

formula $\mathfrak{A}$ is provable in **K**, or, what comes to the same, whether the system consisting of **K** with — $\mathfrak{A}$ adjoined as an extra axiom is consistent.

---

† *Loc. cit.*

[illegible]†[illegible] $\mathfrak{A}$ [illegible] $\mathfrak{A}$ [illegible]- $\mathfrak{A}$ [illegible] **K** [illegible] **K** [illegible] **K** [illegible]- $\mathfrak{A}$ [illegible]

---

† *Loc. cit.*

[illegible]

[illegible]“[illegible]”[illegible]

[illegible]

If the negation of what Gödel has shown had been proved, *i.e.* if, for each $\mathfrak{A}$, either $\mathfrak{A}$ or — $\mathfrak{A}$ is provable, the we should have an immediate solution of the Entscheidungsproblem. For we can invent a machine $\mathfrak{K}$ which will prove consecutively all provable formulae. Sooner or later $\mathfrak{K}$ will reach either $\mathfrak{A}$ or — $\mathfrak{A}$. If it reaches $\mathfrak{A}$, then we know that $\mathfrak{A}$ is provable. If it reaches — $\mathfrak{A}$, then, since **K** is consistent (Hilbert and Ackermann, p. 65), we know that $\mathfrak{A}$ is not provable.

[illegible] $\mathfrak{A}$ [illegible] $\mathfrak{A}$ [illegible]- $\mathfrak{A}$ [illegible] $\mathfrak{K}$ [illegible] $\mathfrak{A}$ [illegible]- $\mathfrak{A}$ [illegible] $\mathfrak{A}$ [illegible] $\mathfrak{A}$ [illegible]- $\mathfrak{A}$ [illegible] **K** [illegible]65[illegible] $\mathfrak{A}$ [illegible]

[illegible] $\mathfrak{K}$ [illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]

Owing to the alsence of integers in **K** the proofs appear somewhat lengthy. The underlying ideas are quite straightforward.

Corresponding to each computing machine 𝓜 we construct a formula Un (𝓜) and we show that, if there is a general method for determining whether Un (𝓜) is provable, then there is a general method for determining whether 𝓜 ever prints 0.

□□**K**□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□𝓜□□□□□□□□Un(𝓜)□□□□□□□□□□□□□□□□□Un(𝓜)□□□□□□□□□□□□□□□𝓜□□□□0□

□□□□□□□□□□□□□“□□□□”□Undecidable□□□□□Un□□□Un(𝓜)□□□□□□□□——□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□“□□”□□□□□□□□□□□□□

□□□m-□□□□□□□

□□0□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□(x, y)□□□□□x□□□□y□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□I□□□□□□□0□1□□□□10□□□□□□□□□□□□□□□□□□□□□□□CCx□□□“□□□□x”□□□□□□□□□□□□□□□□□□□□□□q□□□□□□□□□□□□□

---------- SQUARE Y ----------

CCx	0	1	2	3	4	5	6	7	…
0	q_1								
1	0	q_2							
2	0		q_3						
3	0		1	q_4					
4	0		1		q_1				
5	0		1		0	q_2			
6	0		1		0		q_3		
7	0		1		0		1	q_4	
8	0		1		0		1		
9	0		1		0		1		
…	0		1		0		1		

[illegible] m-[illegible]

[illegible]

The interpretations of the propositional functions involved are as follows:

$R_{S_l}(x,y)$ is to be interpreted as "in the complete configuration x (of $\mathcal{M}$) the symbol on the square y is S".

[illegible]

$R_{S_l}(x, y)$[illegible]"[illegible]$\mathcal{M}$[illegible]x[illegible]y[illegible]S"[illegible]

[illegible]S[illegible]l[illegible]S_0[illegible]S_1[illegible]0[illegible]S_2[illegible]1[illegible]l[illegible]$R_{S_2}(5, 2)$[illegible]1[illegible]5[illegible]2[illegible] $R_{S_2}(5, 6)$[illegible]1[illegible]5[illegible]6[illegible]

I(*x*, *y*) is to be interpreted as "in the complete configuration *x* the square *y* is scanned".

$I(x, y)$□□□□□“□□□□□x□□y□□□□□“□

□□□I□□□□$I(6, 6)$□□□□□□□□□□6□□□□□□□□6□□$I(6, y)$□□□□□□y□□□□□

$K_{q_m}(x)$ is to be interpreted as "in the complete configuration *x* the *m*-configuration is q_m.

$K_{q_m}(x)$□□□□□“□□□□□x□□m-□□□q_m□

□□□□□□□□□□□□□□□□I□□□K_{q_2} (5)□□□□K_{q_3}□K_{q_2}(7)□□□

$F(x, y)$ is to be interpreted as "*y* is the immediate successor of *x*".

$F(x, y)$□□□□□“y□□□x□□□”□

□□□□□□□□□□□□□$y = x + 1$□□□□□□□□□□□□□□□□□□□□$F(u''', u'''')$□□□□□□□□□□□□□□□□□$F(u''', u'''')$□□□□□

□□m-□□q_i□ □□□□□S_j□□□□□□S_k□□□□S_j□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□ q_l□

□□□□□□□□□□□□Inst□□□“□□”□□□m-□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

Inst $\{q_i\ S_j\ S_k\ L\ q_l\}$ is to be an abbreviation for

$$(x,y,x',y')\Big\{\Big(R_{S_j}(x,y)\,\&\,I(x,y)\,\&\,K_{q_i}(x)\,\&\,F(x,x')\,\&\,F(y',y)\Big)$$
$$\rightarrow\Big(I(x',y')\,\&\,R_{S_k}(x',y)\,\&\,K_{q_l}(x')$$
$$\&\,(z)\Big[F(y',z)\vee\Big(R_{S_j}(x,z)\rightarrow R_{S_k}(x',z)\Big)\Big]\Big)\Big\}.$$

□□$\{q_i\ S_j\ S_k\ L\ q_l\}$□□□□□□□□□□□

$$(x,y,x',y')\Big\{\big(R_{S_j}(x,y)\,\&\,I(x,y)\,\&\,K_{q_i}(x)\,\&\,F(x,x')\,\&\,F(y',y)\big)$$
$$\rightarrow \Big(I(x',y')\,\&\,R_{S_k}(x',y)\,\&\,K_{q_l}(x')$$
$$\&\,(z)\Big[F(y',z)\,\mathrm{v}\,\big(R_{S_j}(x,z)\rightarrow R_{S_k}(x',z)\big)\Big]\Big)\Big\}.$$

[illegible] L [illegible]

[illegible] (x, y, x', y') [illegible] $(x)\ (y)\ (x')\ (y')$ [illegible] x [illegible] x' [illegible] F [illegible] x' [illegible] x [illegible] y [illegible] y' [illegible] y' [illegible] y [illegible] 1 [illegible] y [illegible] y' [illegible] F [illegible]

[illegible] x [illegible] y [illegible] S_j [illegible] q_i [illegible]

[illegible] x' [illegible] y' [illegible] y [illegible] S_k [illegible] m-[illegible] q_l [illegible]

[illegible] y [illegible] z [illegible] z [illegible] y' [illegible] z [illegible] y [illegible]…… [illegible] S_j [illegible] S_k [illegible]

[illegible] 295 [illegible]

$$\&(z)[F(y',z)\mathrm{v}([R_{S_0}(x,z)\rightarrow R_{S_0}(x',z)]\,\&\,[R_{S_1}(x,z)\rightarrow R_{S_1}(x',z)]$$
$$\&\cdots\&\,[R_{S_M}(x,z)\rightarrow R_{S_M}(x',z)])])\}$$

R [illegible] $S_0, S_1, \ldots, S_M$ [illegible] $\mathcal{M}$ [illegible] z [illegible] z [illegible] y' [illegible] z [illegible] y —— [illegible] x [illegible] x' [illegible]

[illegible]

Inst $\{q_i\ S_j\ S_k\ R\ q_l\}$ and Inst $\{q_i\ S_j\ S_k\ N\ q_l\}$

are to be abbreviations for other similarly constructed expressions.

[illegible] $\{q_i\ S_j\ S_k\ R\ q_l\}$ [illegible] $\{q_i\ S_j\ S_k\ N\ q_l\}$ [illegible]

[illegible]

[illegible] $\{q_i\ S_j\ S_k\ Rq_l\}$ [illegible]

$$(x, y, x', y') \Big\{ (R_{S_j}(x, y) \,\&\, I(x, y) \,\&\, K_{q_i}(x) \,\&\, F(x, x') \,\&\, F(y, y'))$$

$$\rightarrow (I(x', y') \,\&\, R_{S_k}(x', y) \,\&\, K_{ql}(x')$$

$$\&\, (z)[F(z, y') \vee ([R_{S_0}(x, z) \rightarrow R_{S_0}(x', z)] \,\&$$

$$[R_{S_1}(x, z) \rightarrow R_{S_1}(x', z)] \,\&$$

$$\ldots \&$$

$$[R_{S_m}(x, z) \rightarrow R_{S_m}(x', z)])]) \Big\}$$

□□□□□□□□□□□□□□□□□□□□□□□ $F(y', y)$□□□ $F(y, y')$□□□□□□□□□□□□□□□□□□□ □□□□□□□□□□□□□□ $F(y', z)$□□□ $F(z, y')$□□□□□ z□□ y□

□□□□□I□ R_{S_m}□□□□□ m□□□□□2□□□□□□□□□□ S_0□□□□□ S_1□0□□ S_2□1□□

Let us put the description of $\mathcal{M}$ into the first standard form of §6. This description consists of a number of expressions such as "$q_i\ S_j\ S_k\ L\ q_l$" (or with R or N substituted for L).

□□□□ $\mathcal{M}$ □□□□□§6□□□□□□□□□□□□□□□□□□□□□□□□□□□□□{$q_i\ S_j\ S_k\ L\ q_l$} □□□□ R□ N□□ L□□

□□□□□□□□□5□□□□□□□□□□□□□□□□□241□□□□□126□□□□□I□□□□□

$$q_1 S_0 S_1 R q_2; q_2 S_0 S_0 R q_3; q_3 S_0 S_2 R q_4; q_4 S_0 S_0 R q_1;$$

□□□□□□□□□□□□□□□□□□□□□□

Let us form all the corresponding expressions such as Inst {$q_i S_j S_k L q_l$} and take their logical sum. This we call Des ($\mathcal{M}$).

□□□□□□□□□□□□□□□□□□□□□{$q_i\ S_j\ S_k\ L\ q_l$}□□□□□□□□□□□□□□□□□□Des($\mathcal{M}$)□

□□□□ $\mathcal{M}$ □□□□□□"□□□"□□□□□□□□□□□□□□□□297□□□□□□□"□□"□□□□□□□ □□□□□□□□□□□□□□□□□&□□□□□□□□□□□□□□□□□□□

□□I□Des($\mathcal{M}$)□□□□□□□□□□□

$$\text{Inst}\{q_1 S_0 S_1 R q_2\} \;\&\; \text{Inst}\{q_2 S_0 S_1 R q_3\} \;\&$$
$$\text{Inst}\{q_3 S_0 S_2 R q_4\} \;\&\; \text{Inst}\{q_4 S_0 S_0 R q_1\}$$

□□□□□□□Des($\mathcal{M}$)□□□□□□□□□□Un($\mathcal{M}$)□□□□□□□□□□□□□□□□□□□□□□□□□□ □□

□□□□ → □□0

□□□□□□□□□□□F□□□□□N□□□□□□□□□□□□N□□□

The formula Un($\mathcal{M}$) is to be

$$(\exists u)\Big[N(u)\,\&\,(x)\big(N(x)\rightarrow(\exists x')F(x,x')\big)\\ \&\,(y,z)\big(F(y,z)\rightarrow N(y)\,\&\,N(z)\big)\,\&\,(y)R_{S_0}(u,y)\\ \&\,I(u,u)\,\&\,K_{q_1}(u)\,\&\,\mathrm{Des}(\mathcal{M})\Big]\\ \rightarrow(\exists s)(\exists t)\big[N(s)\,\&\,N(t)\,\&\,R_{S_1}(s,t)\big].$$

□□Un($\mathcal{M}$)□□

$$(\exists u)\Big[N(u)\,\&\,(x)\big(N(x)\rightarrow(\exists x')F(x,x')\big)\\ \&\,(y,z)\big(F(y,z)\rightarrow N(y)\,\&\,N(z)\big)\,\&\,(y)R_{S_0}(u,y)\\ \&\,I(u,u)\,\&\,K_{q_1}(u)\,\&\,\mathrm{Des}(\mathcal{M})\Big]\\ \rightarrow(\exists s)(\exists t)\big[N(s)\,\&\,N(t)\,\&\,R_{S_1}(s,t)\big].$$

□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□s□t□□□□□S_1□0□□□□□□t□□□□□□□s
□□□□□□8□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□0□□□□□□□□□□□□□□□□□□□□□□
□□

□□Un($\mathcal{M}$)□□□□□□□□□u□□□0□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□x□□□□□x□□□x'□

□□□□□□□□□□□□□□□□□□□□u□0□□□□□□□□□y□y□□□□□S_0□□□□□□□□□□□□
□□□□□□□□□□□I□□□□□□□□□□□□□0□□□□□0□□□□□K□□□□□□□□m-□□□□□q_1□□□
□□□□□□□□□□□□□Des($\mathcal{M}$)□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

[N(u)&...& Des($\mathcal{M}$)] many be abbreviated to A($\mathcal{M}$).

[N(u)&...& Des($\mathcal{M}$)]□□□□□A($\mathcal{M}$)□

□□A($\mathcal{M}$)□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□u□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□12□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□G(x, y)□□□y□□x□□□□□□□□□□$\mathbb{Q}$□□□□□□□□□□□□□P□

$\mathcal{Q}$ [illegible]

$$(x)(\exists w)(y,z)\Big\{F(x,w)\ \&\ \big(F(x,y)\rightarrow G(x,y)\big)$$
$$\&\ \big(F(x,z)\ \&\ G(z,y)\rightarrow G(x,y)\big)$$
$$\&\ \Big[G(z,x)\ \mathrm{v}\ \big(G(x,y)\ \&\ F(y,z)\big)\ \mathrm{v}$$
$$\big(F(x,y)\ \&\ F(z,y)\big)\rightarrow\big(-F(x,z)\big)\Big]\Big\}$$

[illegible]$A(\mathcal{M})$[illegible]$\mathcal{Q}$[illegible]

$A(\mathcal{M})$[illegible]

$$\mathcal{Q}\ \&\ (y)R_{S_0}(u,y)\ \&\ I(u,u)\ \&\ K_{q_1}(u)\ \&\ \mathrm{Des}(\mathcal{M})$$

[illegible]$A(\mathcal{M})$[illegible]u[illegible]Un($\mathcal{M}$)[illegible]

Un($\mathcal{M}$)[illegible]

$$(\exists u)A(M)\ \rightarrow\ (\exists s)(\exists t)R_{S_1}(s,t)$$

[illegible]$N(x)$[illegible]R[illegible]I[illegible]K[illegible]F[illegible]Inst[illegible]Des[illegible]

When we substitute the meanings suggested on p. 259-60 we find that Un ($\mathcal{M}$) has the interpretation “in some complete configuration of $\mathcal{M}$, S_1 (*i.e.* 0) appears on the tape”.

[illegible]259[illegible]260[illegible]Un($\mathcal{M}$)[illegible]“[illegible]$\mathcal{M}$[illegible]S_1[illegible]0[illegible]”[illegible]

[illegible]Un($\mathcal{M}$)[illegible]0[illegible]Un($\mathcal{M}$)[illegible]0[illegible]0[illegible]Un($\mathcal{M}$)[illegible]0[illegible]

[illegible]“[illegible]”[illegible]

[illegible]Un($\mathcal{M}$)[illegible]S_1[illegible]1[illegible]2[illegible](a)[illegible](b)[illegible]

Corresponding to this I prove that

(*a*) If S_1 appears on the tape in some complete configuration of $\mathcal{M}$, then Un($\mathcal{M}$) is provable.

(*b*) If Un($\mathcal{M}$) is provable, then S_1 appears on the tape in some complete configuration of $\mathcal{M}$.

When this has been done, the remainder of the theorem is trivial.

[illegible]

(a) [illegible]S_1[illegible]$\mathcal{M}$[illegible]Un($\mathcal{M}$)[illegible]

(b) [illegible]Un($\mathcal{M}$)[illegible]S_1[illegible]$\mathcal{M}$[illegible]

[illegible]

[illegible]1[illegible]

[261]

Lemma 1. *If S_1 appears on the tape in some complete configuration of $\mathcal{M}$, then Un($\mathcal{M}$) is provable.*

We have to show how to prove Un($\mathcal{M}$).

[illegible]1 [illegible]S_1[illegible]$\mathcal{M}$[illegible]Un($\mathcal{M}$)[illegible]

[illegible]Un($\mathcal{M}$)[illegible]

[illegible]u[illegible]0[illegible]u'[illegible]1[illegible]u''[illegible]2[illegible]$u^{(n)}$[illegible]u[illegible]n[illegible]n[illegible]

[illegible]3[illegible]$r(n, m)$[illegible]$i(n)$[illegible]$k(n)$[illegible]n[illegible]m[illegible]

[illegible]$r(n, m)$[illegible]n[illegible]m[illegible]0[illegible]1[illegible]0[illegible]2[illegible]1[illegible]$S_{r(n, m)}$[illegible]n[illegible]m[illegible]r[illegible]S[illegible]R[illegible]

$$R_{S_{r(n,m)}}(n,m)$$

[illegible]u[illegible]R[illegible]

$$R_{S_{r(n,m)}}(u^{(n)},u^{(m)})$$

[illegible]r[illegible]n[illegible]m[illegible]R[illegible]$u^{(n)}$[illegible]$u^{(m)}$[illegible]

[illegible]$i(n)$[illegible]n[illegible]

$$I(u^{(n)},u^{(i(n))})$$

[illegible]n[illegible]$i(n)$[illegible]$k(n)$[illegible]n[illegible]m-[illegible]$q_{k(n)}$[illegible]n[illegible]m-[illegible]

$$K_{q_{k(n)}}(u^{(n)})$$

[illegible]n[illegible]m-[illegible]$q_{k(n)}$[illegible]

Let us suppose that in the n-th complete configuration the sequence of symbols on the tape is $S_{r(n,\,0)}, S_{r(n,\,1)}, \ldots, S_{r(n,\,n)}$, followed by nothing but blanks, and that the scanned symbol is the $i(n)$-th, that the m-configuration is $q_{k(n)}$.

□□□□□□□n□□□□□□□□□□□□□□□□□□$S_{r(n,0)}$, $S_{r(n,1)}$□… $S_{r(n,n)}$, □□□□□□□□□
□□□□□□□$i(n)$□□m-□□□$q_{k(n)}$□

□□□□□□0□□□□□□□□□□□□□□□□□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
n□□□□n□□□□□□□□□□□□□□□□□□□□□□□□□□□0□□□□□□□□□□□□$S_{r(n,0)}$, $S_{r(n,1)}$,…
$S_{r(n,n)}$□□□□□□□□□□□$n+1$□□□□□□□n□□□□□□□□□□□□□□□r□□□□□□□□0□□□□□□□□□
□□

Then we may form the proposition

$$\begin{aligned}
&R_{S_{r(n,\,0)}}(u^{(n)}, u) \;\&\; R_{S_{r(n,\,1)}}(u^{(n)}, u') \;\&\; \ldots \;\&\; R_{S_{r(n,\,n)}}(u^{(n)}, u^{(n)}) \\
&\qquad \& \; I\left(u^{(n)}, u^{(i(n))}\right) \;\&\; K_{q_{k(n)}}(u^{(n)}) \\
&\qquad \& \; (y)F\left((y, u') \vee F(u, y) \vee F(u', y) \vee \ldots \vee F(u^{(n-1)}, y) \vee R_{S_0}(u^{(n)}, y)\right),
\end{aligned}$$

which we may abbreviate to CC_n.

□□□□□□□□□□□□□□□□

$$\begin{aligned}
&R_{S_{r(n,\,0)}}(u^{(n)}, u) \;\&\; R_{S_{r(n,\,1)}}(u^{(n)}, u') \;\&\; \ldots \;\&\; R_{S_{r(n,\,n)}}(u^{(n)}, u^{(n)}) \\
&\qquad \& \; I\left(u^{(n)}, u^{(i(n))}\right) \;\&\; K_{q_{k(n)}}(u^{(n)}) \\
&\qquad \& \; (y)F\left((y, u') \vee F(u, y) \vee F(u', y) \vee \ldots \vee F(u^{(n-1)}, y) \vee R_{S_0}(u^{(n)}, y)\right),
\end{aligned}$$

□□□□□□□□CC_n□

□□□□□□“□□□□n”□□□□□□□□□□□□$n+1$□□□□□□□□□□□□□□□□□□□□□□□□□□□
$i(n)$□m-□□$q_k\,(n)$□□□□

□□□□□□□□□□□□□□□□□□F□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□0□□n□□□□□□
□□□□□□□□□□□□n□□□□□□□□□R□□□□□□□□□□□□□□y□□□□□□□□□□□□□□□u'□y□□□□
□□y□0□□□□y□□u□□□□□□y□1□□□□y□u'□□□□□□y□2□□□□□□□□□□□y□n-1□□□
□□y□n□□□□y□□□□□□□□□□□□□□□□□y□□□□□□□□□
□□□□□□□□□□□

CC_n□□□□□□□□□□□

$$R_{S_{r(n,0)}}(u^{(n)},u) \mathbin{\&} R_{S_{r(n,1)}}(u^{(n)},u') \mathbin{\&} \dots \mathbin{\&} R_{S_{r(n,n)}}(u^{(n)},u^{(n)})$$

$$\mathbin{\&} I(u^{(n)},u^{(i(n))}) \mathbin{\&} K_{q_{k(n)}}(u^{(n)})$$

$$\mathbin{\&} (y)\Big(F(y,u') \vee F(u,y) \vee F(u',y) \vee \dots \vee F(u^{(n-1)},y) \vee R_{S_0}(u^{(n)},y)\Big),$$

□n□□0□□□□□□□□□□□□□□□□□□□CC_0□□□□□□□□□□□

$$I(u,y) \mathbin{\&} K_{q_1}(u) \mathbin{\&} (y)R_{S_0}(u,y)$$

As before, $F(u, u')$ & $F(u', u'')$ &...& $F(u^{(r-1)}, u^{(r)})$ is abbreviated to $F^{(r)}$.

I shall show that all formulae of the form $A(\mathcal{M})$ & $F^{(n)} \to CC_n$ (abbrevated to CF_n) are provable.

□□□□□□ $F(u, u')$&$F(u', u'')$&...&$F(u^{(r-1)},u^{(r)})$ □□□$F^{(r)}$□

□□□□□□□□$A(\mathcal{M})$ & $F^{(n)} \to CC_n$□□□□CF_n□□□□□□□□□□□□

□□$A(\mathcal{M})$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□q_1□□□□0□□□□□□Des($\mathcal{M}$)□□□□□□□□□□□□□□Des($\mathcal{M}$)□□□m-□□□□□□□□□□□□□□□□

□□□□□□□□□□n□□□□□□□□CF_n□□□□

CF_n□□□□□□□□□□

$$A(\mathcal{M}) \mathbin{\&} F^{(n)} \to CC_n$$

□□□CF_n□□□□□□□□

$$CF_0\colon A(\mathcal{M}) \to CC_0$$

$$CF_1\colon A(\mathcal{M}) \mathbin{\&} F(u,u') \to CC_1$$

$$CF_2\colon A(\mathcal{M}) \mathbin{\&} F(u,u') \mathbin{\&} F(u',u'') \to CC_2$$

$$CF_3\colon A(\mathcal{M}) \mathbin{\&} F(u,u') \mathbin{\&} F(u',u'') \mathbin{\&} F(u'',u''') \to CC_3$$

...

The meaning of CF_n is “The n-th complete configuration of $\mathcal{M}$ is so and so” where “so and so” stands for the actual n-th complete configuration of $\mathcal{M}$. That CF_n should be probable is therefore to be expected.

CF_n□□□□“$\mathcal{M}$□□n□□□□□□□□”□□□“□□” □□□$\mathcal{M}$□□□□n□□□□□□□□□□ □□□□CF_n□□□□□□

□□□□□□□□□CF_n□□□□□□□□□□□□□CF_0□□□□□□□□□□□CF_n□□□□□□□ CF_{n+1}□□□□□□□

CF_0 is certainly provable, for in the complete configuration the symbols are all blanks the m-configuration is q_1, and the scanned square is u, *i.e.* CC_0 is

$$(y)\, R_{S_0}(u,y)\ \&\ I(u,u)\ \&\ K_{q_1}\ (u).$$

CF_0□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□q_1□□□□□u□□CC_0□

$$(y)\, R_{S_0}(u,y)\ \&\ I(u,u)\ \&\ K_{q_1}\ (u).$$

□□□□CC_0□□□□□□□□□□□□□□□□□□□$A(\mathcal{M})$□□□□□

$$Q\ \&\ (y)R_{S_0}(u,y)\ \&\ I(u,u)\ \&\ K_{q_1}(u)\ \&\ \mathrm{Des}(\mathcal{M})$$

$A(\mathcal{M})$□□□□CC_0□□□□□□□R□I□K□

$A(\mathcal{M}) \to CC_0$ is then trivial.

We next show that $CF_n \to CF_{n+1}$ is provable for each *n.*

□□□□$A(\mathcal{M}) \to CC_0$□

□□□□□□□□□n□ $CF_n \to CF_{n+1}$□□□□□

□□□CF_n□□□□□□□□

$$CF_n \to CF_{n+1}$$

□□□□□□□□□

$$\left[A(\mathcal{M})\ \&\ F^{(n)} \to CC_n\right] \to \left[A(\mathcal{M})\ \&\ F^{(n+1)} \to CC_{n+1}\right]$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$CF_0 \to CF_1$□□□□□□$CF_1 \to CF_2$□□□□□□□□□□□□□□□CF_n□□□□□□□□□

There are three cases to consider, according as in the move from the n-th to the $(n + 1)$-th configruation the machine moves to left or to right or remains stationary. We suppose that the first case applies, *i.e.* the machine moves to the left. A similar argument applies in the other cases.

□□□n□□□□□□n+1□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□ □□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□r□i□k□□□□a□b□c□d□□□□□□□□□□□□□□□

If $r(n, i(n)) = a$, $r(n + 1, i(n+1)) = c$, $k(i(n)) = b$, and $k(i(n + 1)) + d$,

□□
$r(n, i(n))=a$□$r(n+1, i(n+1))=c$□$k(i(n))$□b□□$k(i(n+1))$□d□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□n□□□

$a = k(n)$□m-□□□□□

$b = r(n, i(n))$□□□□$i(n)$□□□□□□

$c = k(n+1)$□□□□m-□□□□□

$d = r(n+1, i(n))$□□□□$i(n)$□□□□□□□

a□c□q□□□□b□d□□S□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

then Des $\mathcal{M}$ must include Inst $\{q_a\ S_b\ S_d\ L\ q_c\}$ as one of its terms, *i.e.*
Des$\mathcal{M}$) → Inst$\{q_a\ S_b\ S_d\ L\ q_c\}$.

□□Des($\mathcal{M}$)□□□□□□□$\{q_a S_b S_d L q_c\}$□□□□□□□□□
Des($\mathcal{M}$) → Inst$\{q_a\ S_b\ S_d\ L\ q_c\}$

□□Des□□□Inst□□□□□□□□□□□□□Des□□□□□□□□□□Inst□□□□□□□□□□□□□□□□□□$A(\mathcal{M})$□Des□□□□□□□□□□□□□□

$$A(\mathcal{M}) \to \text{Des}(\mathcal{M})$$

□□□□□□□□□□□□□□□□□□□□□

$$A(\mathcal{M}) \to \text{Inst}\{q_a S_b S_d L q_c\}$$

□□□$F^{(n+1)}$□F□□□□□□□□□□□F□□□□□□□□□□□□□□□$F^{(n+1)}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□

Hence
$$A(\mathcal{M})\ \&\ F^{(n+1)} \to \text{Inst}\{q_a S_b S_d L q_c\}\ \&\ F^{(n+1)}.$$
But $\text{Inst}\{q_a S_b S_d L q_c\}\ \&\ F^{(n+1)} \to (CC_n \to CC_{n+1})$
is provable,

□□ $A(\mathcal{M})\ \&\ F^{(n+1)} \to \text{Inst}\{q_a S_b S_d L q_c\}\ \&\ F^{(n+1)}$。
□ $\text{Inst}\{q_a S_b S_d L q_c\}\ \&\ F^{(n+1)} \to (CC_n \to CC_{n+1})$□□□□□,

□□□□□□□□□□□□□□□□□□□□□□ $\mathcal{Q}$ □□□□□□□□□□□□□□□

$$\text{Inst}\{q_a S_b S_d L q_c\}\ \&\ \mathcal{Q}\ \&\ F^{(n+1)} \to (CC_n \to CC_{n+1})$$

□□□Inst□□□□□□□a□b□c□d□□□□□□□□□□□□□□Inst□□□□□□□□□□□n□□□□□□$(n+1)$□

$$\mathrm{Inst}\,\{q_{k(n)}S_{r(n,i(n))}S_{r(n+1,i(n))}Lq_{k(n+1)}\}\ \&\ Q\ \&\ F^{(n+1)} \rightarrow (CC_n \rightarrow CC_{n+1})$$

□□□□□□

$$(CC_n\ \&\ \mathrm{Inst}\,\{q_{k(n)}S_{r(n,i(n))}S_{r(n+1,i(n))}Lq_{k(n+1)}\}\ \&\ Q\ \&\ F^{(n+1)}) \rightarrow CC_{n+1}$$

□□□□□□□□□□□Inst□□□CC_n□□□CC_{n+1}□□CC_{n+1}□□□CC_n□□□Inst□CC_n□□□□□□□

□□□□□□□□□□□□□□□□□□□□

and so therefore is

$$A(\mathcal{M})\ \&\ F^{(n+1)} \rightarrow (CC_n \rightarrow CC_{n+1})$$

□□

$$A(\mathcal{M})\ \&\ F^{(n+1)} \rightarrow (CC_n \rightarrow CC_{n+1})$$

□□X → (Y → Z)□□□□□□□□□□□□□□□X → (Y → Z)□□□□□□

$$\left(A(\mathcal{M})\ \&\ F^{(n+1)} \rightarrow CC_n\right) \rightarrow \left(A(\mathcal{M})\ \&\ F^{(n+1)} \rightarrow CC_{n+1}\right)$$

□□□□□□□□□F□F□□□□□□□□□□□$F^{(n+1)} \rightarrow F^{(n)}$□□

$$(A(\mathcal{M})\ \&\ F^{(n+1)}) \rightarrow (A(\mathcal{M})\ \&\ F^{(n)})_{\circ}$$

[262]

and

$$\left(A(\mathcal{M})\ \&\ F^{(n)} \rightarrow CC_n\right) \rightarrow \left(A(\mathcal{M})\ \&\ F^{(n+1)} \rightarrow CC_{n+1}\right),$$

□□

$$\left(A(\mathcal{M})\ \&\ F^{(n)} \rightarrow CC_n\right) \rightarrow \left(A(\mathcal{M})\ \&\ F^{(n+1)} \rightarrow CC_{n+1}\right),$$

□□□□□□□□□□□□□□□□CF_n□□□□□

i.e.

$$CF_n \rightarrow CF_{n+1}.$$

CF_n is probable for each n.

□$CF_n \rightarrow CF_{n+1}$□

□□□□n□CF_n□□□□□□□

□□□□□□□□CF_n□□□Un($\mathcal{M}$)□

Now it is the assumption of this lemma that S_1 appears somewhere, in some complete configuration, in the sequence of symbols printed by $\mathcal{M}$; that is, for some integers N, K,

[illegible]S_1[illegible]$\mathcal{M}$[illegible]N[illegible]K[illegible]

[illegible]N[illegible]K[illegible]

CC_N has R_{S_1} $(u^{(N)}, u^{(K)})$ as one of its terms, and therefore $CC_N \to R_{S_1}$ $(u^{(N)}, u^{(K)}$ is provable.

R_{S_1} $(u(^{N)}, u^{(K)}$ [illegible] CC_N [illegible]$CC_N \to R_{S_1}$ $(u^{(N)}, u^{(K)})$[illegible]

[illegible]

We have then

$$CC_N \to R_{S_1}(u^{(N)}, u^{(K)})$$

and $A(\mathcal{M})$ & $F^{(N)} \to CC^N$.

[illegible]

$$CC_N \to R_{S_1}(u^{(N)}, u^{(K)})$$

[illegible] $A(\mathcal{M})$ & $F^{(N)} \to CC^N$。

CC[illegible]CF_N[illegible]CF_N[illegible]N[illegible]n[illegible]N[illegible]

[illegible]u[illegible]

We also have

$$(\exists u)A(\mathcal{M}) \to (\exists u)(\exists u') \ldots (\exists u^{(N')})\big(A(\mathcal{M}) \ \& \ F^{(N)}\big),$$

where $N' = \max(N, K)$.

[illegible]

$$(\exists u)A(\mathcal{M}) \to (\exists u)(\exists u') \ldots (\exists u^{(N')})\big(A(\mathcal{M}) \ \& \ F^{(N)}\big),$$

[illegible] $N' = \max(N$[illegible]$K)$[illegible]

[illegible]K[illegible]0[illegible]N[illegible]N'[illegible]N[illegible]$A(\mathcal{M})$ & $F^{(N)}$[illegible]CC_N[illegible]CC_N[illegible] R_{S_1} $(u^{(N)}, u^{(K)})$[illegible]

And so

$$(\exists u)A(\mathcal{M}) \to (\exists u)(\exists u') \ldots (\exists u^{(N')})R_{S_1}(u^{(N)}, u^{(K)}),$$

[illegible]

$$(\exists u)A(\mathscr{M}) \to (\exists u)(\exists u') \dots (\exists u^{(N')})R_{S_1}(u^{(N)}, u^{(K)}),$$

R□□□□□□□u□$u^{(N')}$□□□□□□□□□□□□□□$u^{(N)}$□$u^{(K)}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$(\exists u)A(\mathscr{M}) \to (\exists u^{(N)})(\exists u^{(K)})R_{S_1}(u^{(N)}, u^{(K)}),$$

$$(\exists u)A(\mathscr{M}) \to (\exists u^{(N)})(\exists u^{(K)})R_{S_1}(u^{(N)}, u^{(K)}),$$

□□□□□□s□□$u^{(N)}$□t□□$u^{(K)}$□□□□

$$(\exists u)A(\mathscr{M}) \to (\exists s)(\exists t)R_{S_1}(s, t),$$

$$(\exists u)A(\mathscr{M}) \to (\exists s)(\exists t)R_{S_1}(s, t),$$

□□□□□□Un($\mathscr{M}$)□□□□□□□□□□254□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□Un($\mathscr{M}$)□□□□□□□□□$N(s)$□$N(t)$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□s□t□□□□□□□□□□□□□□□□□□□□□□□□□□

□“□□S_1□□□$\mathscr{M}$□□□□□□□□□□□□□□□”□□□□□□□□□□□□□□□□□Un($\mathscr{M}$)□

i.e. Un($\mathscr{M}$) is provable.

This completes the proof of Lemma 1.

□Un($\mathscr{M}$)□□□□□□

□□□□□□□□□1□□□□

□□□

LEMMA 2. *If* Un($\mathscr{M}$) *is provable, then* S_1 *appears on the tape in some complete configuration of* $\mathscr{M}$.

If we substitute any propositional functions for function variables in a provable formula, we obtain a true proposition. In particular, if we substitute the meanings tabulated on pp. 257-260 in Un ($\mathscr{M}$), we obtain a true proposition with the meaning “S_1 appears somewhere on the tape in some complete configuration of $\mathscr{M}$".

□□2 □□Un($\mathscr{M}$)□□□□□□□□□S_1□□□M□□□□□□□□□□□□□□□□

□□“S_1□□□$\mathscr{M}$□□□□□□□□□□□□□□□”□□□259~260□□Un($\mathscr{M}$)□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□Un($\mathscr{M}$)□□□□□□□□ S_1□□□□□$\mathscr{M}$□□□□□□□□□□□□□□□□□

We are now in a position to show that the Entscheidungsproblem cannot be solved. Let us suppose the

contrary. Then there is a general (mechanical) process for determining whether Un($\mathcal{M}$) is provable. By Lemmas 1 and 2, this implies that there is a process for determining whether $\mathcal{M}$ ever prints 0, and this is impossible, by § 8. Hence the Entscheidungsproblem cannot be solved.

[illegible]Un($\mathcal{M}$)[illegible]1[illegible]2[illegible]$\mathcal{M}$[illegible]0[illegible]§8[illegible]

[illegible]

Un($\mathcal{M}$)[illegible]Un($\mathcal{M}$)[illegible]A($\mathcal{M}$)[illegible]A($\mathcal{M}$)[illegible]Q[illegible]Des($\mathcal{M}$)[illegible]Des($\mathcal{M}$)[illegible]Inst[illegible]Inst[illegible]Q[illegible]A($\mathcal{M}$)[illegible]Un($\mathcal{M}$)[illegible]

[illegible]1915[illegible]·[illegible]1878—1957[illegible][2]

[illegible]“[illegible]”[illegible]

[illegible][3][illegible]$\forall$[illegible]$\exists$[illegible]

1928[illegible]·[illegible]·[illegible]1889—1942[illegible]$\exists^*\forall^*$[illegible]1928[illegible]·[illegible]$\exists^*\forall\exists^*$[illegible]1932[illegible]$\exists^*\forall^2\exists^*$[illegible]

[illegible]1920[illegible]$\forall^*\exists^*$[illegible]1933[illegible]$\forall^3\exists^*$[illegible]

[illegible]1932[illegible]$\exists^*\forall^2\exists^*$[illegible]$\forall^2\exists^*$[illegible]$\forall^3\exists^*$[illegible]$\forall^2\exists^*$[illegible]$\forall^3\exists^*$[illegible]

[illegible]quantor[illegible]

In view of the large number of particular cases of solutions of the Entscheidungsproblem for formulae with restricted systems of quantors, it

[263]

is interesting to express Un($\mathscr{M}$) in a form in which all quantors are at the beginning. Un($\mathscr{M}$) is, in fact, expressible in the form

$$(u)(\exists x)(w)(\exists u_1)\ldots(\exists u_n)\mathfrak{B}, \tag{I}$$

where $\mathfrak{B}$ contains no quantors, and $n = 6$.

[illegible]quantor[illegible]Un($\mathscr{M}$)[illegible]Un($\mathscr{M}$)[illegible]

$$(u)(\exists x)(w)(\exists u_1)\ldots(\exists u_n)\mathfrak{B}, \tag{I}$$

[illegible]$\mathfrak{B}$[illegible]$n = 6$[illegible]

[illegible] $\forall\exists\forall\exists^6$ [illegible]

By unimportant modifications we can obtain a formula, with all essential properties of Un($\mathscr{M}$), which is of form (I) with $n = 5$.

[illegible]Un($\mathscr{M}$)[illegible](I)[illegible]$n = 5$[illegible]

[illegible]4[illegible]$\forall\exists\forall\exists^4$[illegible]

1962[illegible]·[illegible]·[illegible]1924—1984[illegible][4] [illegible] $\exists \& \forall\exists\forall$ [illegible] $\forall\exists\forall$ [illegible] $\forall\exists\forall$ [illegible]

[illegible]n[illegible]2[illegible]$x^n + y^n = z^n$ [illegible]

[illegible]

[1] □□□□□□□□□□*Grundzüge der Theoretischen Logik*□Springer□1928□□73□77□

[2] □□□□·□□□□□“Über Möglichkeiten im Relativekalkül”□*Mathematische Annalen*□Vol. 76□1915□□447-470□□□□“On Possibilities in the Calculus of Relatives”□□□Jean van Heijenoort□ed.□*From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931*□Harvard University Press□1967□□228-251□

[3] □□□□□□Egon Börger□Erich Grädel□Yuri Gurevich□*The Classical Decision Problem*□Springer□1997□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

[4] J. Richard Büchi□“Turing-Machines and the Entscheidungsproblem”□*Mathematische Annalen*□Vol. 148□No.3□June 1962□□201-213□

[illegible]15[illegible]
λ [illegible]

[illegible]1983[illegible]1984[illegible]·[illegible]80[illegible]Xerox Dandelion[illegible]LISP[illegible]·[illegible]1927—2011[illegible]LISP[illegible]50[illegible]*λ*[illegible]

[illegible][1] [illegible]·[illegible]

λ[illegible]·[illegible]20[illegible]30[illegible]

[illegible]*λ*[illegible]*λ*[illegible]ALGOL[2][illegible]Pascal[illegible]C[illegible]C[illegible] C++[illegible]Java[illegible]C#[illegible]

λ[illegible]LISP[illegible]APL[illegible]Haskell[illegible]Scheme[illegible]F#[illegible]

[illegible]·[illegible]1903[illegible]24[illegible]1929[illegible]1967[illegible]UCLA[illegible]1990[illegible]

[illegible]Duco[illegible]Duco[illegible][3]

[illegible]31[illegible]·[illegible]1931[illegible]·[illegible]·[illegible]1934[illegible]·[illegible]1947[illegible]·[illegible]1950[illegible]·[illegible]1952[illegible]·[illegible]1959[illegible]·[illegible]1938[illegible][4]

[illegible]

[illegible]1927[illegible]1929[illegible]*λ*[illegible]

[illegible]1931[illegible]10[illegible]5[illegible]4[illegible][5] [illegible] lambda[illegible]*λ*[illegible]

[illegible]

$$x^2+5x+7$$

[illegible]

$$f(x)=x^2+5x+7$$

[illegible] x [illegible]

$$f(y)=y^2+5y+7$$

[illegible] $f(4)$ [illegible] 4 [illegible] x [illegible]

$$f(4)=(4)^2+5\cdot(4)+7=43$$

[illegible] y^2+5y+7 [illegible] y [illegible] 4 [illegible] y [illegible]

$$[y^2+5y+7](4)$$

[illegible]

$$[y^2+5y+18x-2xy+7](4)$$

[illegible]

$$[y^2+5y+18x-2xy+7](4,5)$$

[illegible] x [illegible] y [illegible]

[illegible] λy。[illegible] ^y [illegible] lambda[6] [illegible] λy [illegible]

[illegible]

$$\lambda x[M]$$

[illegible] M [illegible] x [illegible]

$$\lambda x[x^2+5x+7]$$

[illegible]

$$\{F\}(A)$$

F [illegible] F [illegible] A [illegible] x [illegible]

$$\{\lambda x[M]\}(A)$$

[illegible] M [illegible]

$$\{\lambda x[x^2+5x+7]\}(A)$$

[illegible] x [illegible] 4 [illegible]

$$\{\lambda x[x^2+5x+7]\}(4)$$

[illegible]

[illegible]

$$\{\{F\}(A)\}(B)$$

[illegible]

$$\{F\}(A,B)$$

[illegible] F [illegible]

$$\{\lambda x\lambda y[y^2+5y+18x-2xy+7]\}(A,B)$$

[illegible] A [illegible] x [illegible] B [illegible] y [illegible] λ [illegible]

[illegible]

$$\{F\}(A,B)$$

[illegible]

$$F(A,B)$$

[illegible] F [illegible] λ [illegible]

$$\lambda x\lambda y[M](A,B)$$

[illegible] λ [illegible]

$$\lambda x\lambda y.M(A,B)$$

[illegible] lambda [illegible]

[illegible] lambda [illegible]

I [illegible] x [illegible] y [illegible]

II [illegible] $\{\lambda x.M\}(N)$ [illegible] N [illegible] x [illegible] N [illegible] M [illegible] x [illegible] N [illegible] x [illegible] M [illegible]

III [illegible] II [illegible]

[illegible] lambda [illegible] [7] [illegible] “[illegible]” [8] [illegible]

[illegible] conv [illegible] “[illegible]” [illegible] I [illegible] II [illegible] III [illegible]

$$\lambda x[x^2+5x+7](A)\,\mathrm{conv}\,A^2+5A+7$$

[illegible] lambda [illegible] 1 [illegible] 5 [illegible] “[illegible]” [9]

[illegible] · [illegible] · [illegible] · [illegible] 1907—1989 [illegible] 1934 [illegible] ” [illegible] ” [10] [illegible] lambda [illegible]

$$\lambda x\lambda yM$$

[illegible]

$$\lambda xyM$$

[illegible]

[illegible] 1935 [illegible] “[illegible]” [illegible] [11] [illegible] [12]

[illegible] λ [illegible]

□□□□□□□□□□□□□□□0□1□□□□□□□□□□□□1□□□□□□□□□□□□□[13]

$$1 \to \lambda fx.f\left(x\right)$$

□□□□□□□□□“□□”□“□□□”□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$1 \to \left\{\lambda fx\left[f\left(x\right)\right]\right\}$$

□□□1□□□□□□□□□□□□□□□f□x□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□

$$S \to \lambda\rho fx.f\left(\rho\left(f,x\right)\right)$$

□□3□□□□□□
□□□□□□□□2□□□□□□□□□□□□□

$$2 \to S\left(1\right)$$

□□□□□□□□□□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□1□□□□□□□□

$$1 \to \lambda ab.a\left(b\right)$$

□□□λ□□
□□□□□□□□□□□□□□□□□□□□□

□□S(1)□□□□□{S}1□□□

$$\left\{\lambda\rho fx.f\left(\rho\left(f,x\right)\right)\right\}\left(\lambda ab.a\left(b\right)\right)$$

□□□□□□□□□□□□□ρ□□□□□1□□□□□□□□□□ρ□□□□λ□□ρ□□□□□□

$$\lambda fx.f\left(\lambda ab.a\left(b\right)\left(f,x\right)\right)$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$\lambda fx.f\left(\left\{\lambda ab.a\left(b\right)\right\}\left(f,x\right)\right)$$

□f□□a□x□□b□

$$\lambda fx.f\left(f\left(x\right)\right)$$

□□□□□□□□

□□1□□□□□□□

$$1 \to \lambda fx.f\left(x\right)$$

□□2□□□□

$$2 \to S\left(1\right)\mathrm{conv}\lambda fx.f\left(f\left(x\right)\right)$$

□□□□□□2□1□□□□□□□□□□□□□□□□□□*f*□□□□□□□□□□□□□□□□*λab.a*(*a*(*b*))□□□2□□□□□□□□□□□□{*S*}(2):

$$\{\lambda\rho fx.f(\rho(f,x))\}(\lambda ab.a(a(b)))$$

□□□□2□□□*ρ*□

$$\lambda fx.f(\{\lambda ab.a(a(b))\}(f,x))$$

□*f*□□*a*□*x*□□*b*□

$$\lambda fx.f(f(f(x)))$$

□□3□*λ*□□□

□□□□□“□”□□□□

$$+\rightarrow\lambda\rho\sigma fx.\rho(f,\sigma(f,x))$$

□□□□□□□□□□□□2□3□□□□□□□□□□□□□□□□□□□□□□□□□*λab.a*(*a*(*b*))□□2□*λcd.c*(*c*(*c*(*d*)))□□3□□□{+}(2,3)□□□

$$\{\lambda\rho\sigma fx.\rho(f,\ \sigma(f,x))\ \}(\lambda ab.a(a(b)),\lambda cd.c(c(c(d))))$$

□+□□□□□2□□□□□*ρ*□□3□□□□□*σ*□

$$\lambda fx.\lambda ab.a(a(b))(f,\{\lambda cd.c(c(c(d)))\}(f,x))$$

□□□□3□□□□□□□□□*f*□□*c*□*x*□□*d*□

$$\lambda fx.\{\lambda ab.a(a(b))\}(f,f(f(f(x))))$$

□□□□□□2□□□□*f*□□*a*□□*f*(*f*(*f*(*x*)))□□*b*□□□□

$$\lambda fx.f(f(f(f(f(x)))))$$

□□□□□□□□□□□□□□□□□*S*(*S*(*S*(*S*(1))))-□□□□□5□

□□□□□□□□□□□□□□□□□□□□□

$$\times\rightarrow\lambda\rho\sigma x.\rho(\sigma(x))$$

□□□□□2□3□□□□□□{×}(2,3)□□□

$$\{\lambda\rho\sigma x.\rho(\sigma(x))\}(\lambda ab.a(a(b)),\lambda cd.c(c(c(d))))$$

□2□□□□□*ρ*□3□□□□□*σ*□

$$\lambda x.\lambda ab.a(a(b))(\{\lambda cd.c(c(c(d)))\}(x))$$

□□3□□□□□x□□c□□□□

$$\lambda x.\{\lambda ab.a(a(b))\}(\lambda d.x(x(x(d))))$$

□□2□□□□□□□□□□□□□□□□a□

$$\lambda x.\lambda b\lambda d.x(x(x(d)))(\{\lambda d.x(x(x(d)))\}(b))$$

□□□□□□□□b□□d

$$\lambda x.\lambda b.\{\lambda d.x(x(x(d)))\}(x(x(x(b))))$$

□□□□d□

$$\lambda xb.x(x(x(x(x(x(b))))))$$

□□□6□□□□□2□3□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□

{2}(3□

□□

$$\{\lambda ab.a(a(b))\}(\lambda cd.c(c(c(d))))$$

□□□□□□□□□□□□□□□□□□□□

$$\lambda bd.b(b(b(b(b(b(b(b(b(d)))))))))$$

□□□9□□□□□□□□□3□□□□□□□□□□□□m□n□□□□□□

$$\lambda mn.nm$$

□□λ□□□□□□□□□□λ□□□□□□□□□□□□□□□□□□□□□□□□λ□□□□□□□□“□□□□□□□□□□□□□□□□□λ□□□□□□□□□□□□□□□□□”[14]

□□□·□□□□1933□□□□□□□□□1934□□[15] □□□□□□□□□□□□□□□·□□□□□□1908—1931□□1931□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□λ□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

1936□□□□□□□“□□□□□□□□□□”[16]□□□□□□□□□□□□“λ□□□□”□□□□□□□□□□□□□lambda□□□□□□□□□□□□□□□□□“□□□□”□□“□□□□□□”□□λ□□□□□□□□□□□□□[17]□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□“□□□□□□□□”□□
□□□□□□□“□□□□□□□□□□□□□□□”[18]□□□□□□1936□4□15□□□□□□□□□□□
□1936□5□28□□□□□□□□□□□□□□□6□□

1936□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□λ□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□8□28□□□□□□□□□□□□□□□□□□□“□□□□□□□□□
□□□□□□□□”□□□□□□□□□□□9□23□□□□□□□□□□□□□□29□□□□□□[19]

Added 28 *August,* 1936.

APPENDIX.

Computability and effective calculability

The theorem that all effectively calculable (λ-definable) sequences are computable and its converse are proved below in outline.

1936□8□28□□□

□□

□□□□□□□□□□□

□□□□□□□□□λ□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□“□□”□□□□□□□□□□□□□□□

It is assumed that the terms “well-formed formula”□W.F.F.) and “conversion” as used by Church and Kleene are understood. In the second of these proofs the existence of several formulae is assumed without proof; these formulae may be constructed straightforwardly with the help of, *e.g.*, the results of Kleene in “A theory of positive integers in formal logic”, *American Journal of Math.*, 57 (1935), 153-173, 219-244.

□□□□□□□□□□□□□□□□□□“□□□□”□W.F.F.□□“□□”□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□“□□□□□□□□□□”[□□□□□□□□
□□□57□1935□□153-173□219-244]□□□□□

□□□□□□“□□□□□”□□□□□□□□□□□□□□□□□□□□□□□λ□□□□□

The W.F.F. representing an integer n will be denoted by N_n.

□□□□□□n□W.F.F.□□□N_n□

□□□□□□□□□□□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□N_1□□$\lambda xy.x(y)$, N_2□$\lambda xy.x$□$x(y))$, N_n□$\lambda xy.x(x(x(x\ldots(y)\ldots)))$.

We shall say that a sequence γ whose n-th figure is $\varphi_\gamma(n)$ is λ-definable or effectively calculable if $1 + \varphi_\gamma(u)$ is a λ-definable function of n,

□□$1+\varphi_\gamma(u)$□n□λ□□□□□□□□□□□□n□□□□$\varphi_\gamma(n)$□□□γ□λ□□□□□□□□□□□□□□□□

φ_r□□□□□□□□□□□□n□□□□u□□□□□□□□□$1+\varphi_\gamma(n)$□□□□□□γ□□n□□0□□1□□□□□□□□□□□λ□□□□□□□□□□□□□□□0□$\varphi_\gamma(n)$□□□λ□□□□□□□0□□λ□□□□□□□□□□□□□1□□1□2□

i.e. if there is a W.F.F.

M_γ such that, for all integers n,

$$\{M_\gamma\}(N_n) \text{ conv } N_{\varphi\gamma(n)+1,}$$

i.e.$\{M_\gamma\}(N_n)$ is convertible into $\lambda xy.x(x(y))$ or into $\lambda xy.x(y)$ according as the n-th figure of λ is 1 or 0.

□□□□W.F.F. M_γ□□□□□□□□□□□n□

$$\{M_\gamma\}(N_n)\text{conv}N_{\varphi_\gamma(n)+1,}$$

□□□λ□□n□□□1□□0□$\{M_\gamma\}(N_n)$□□□□□$\lambda xy.x(x(y))$□□□$\lambda xy.x(y)$.

□□□□□λ□□□□□□□□□"γ□□n□□"□□□□□□□□□1□□0□□N_n□□□□□□□□□□□□□□M_γ□□□□ $\lambda xy.x(x(y))$□□□□2□□□ $\lambda xy.x(y)$□□□□1□□

□□□□□γ□□5□□1□□□$\varphi_{\gamma(5)}$□1□□□

$$\{M_\gamma\}(N_5)\text{ conv}N_{\varphi\gamma}(5)+1$$

□□□□

$$\{M_\gamma\}(N_5)\text{ conv}N_2$$

To show that every λ-definable sequence γ is computable, we have to show how to construct a machine to compute γ. For use with machines it is convenient to make a trivial modification in the calculus of conversion. This alteration consists in using x, x', x'', … as variables instead of a, b, c, ….

□□□□□□□□λ□□□□□γ□□□□□□□□□□□□□□□□□□□□□□□□γ□□x, x', x'', … □□□□□□□□□□□□a, b, c….

□□□□□□□□□□□□a□b□c□□□□□□□□□□□□□x□y□□□8□□□□□□□□□□□□205□□□□□□□□□□□□□□□□□□□□□"□□□"□

We now construct a machine $\mathscr{L}$ which, when supplied with the formula M_γ, writes down the sequence γ. The construction of $\mathscr{L}$ is some-what similar to that of the machine $\mathscr{K}$ which proves all provable formulae of the functional calculus. We first construct a choice machine $\mathscr{L}_1$, which, if supplied with a W.F.F., M say, and suitably manipulated, obtains any formula into which M is convertible. $\mathscr{L}_1$ can then be modified so as to yield

an automatic machine $\mathscr{L}_2$ which obtains successively all the formulae

[264]

into which M is convertible (cf. foot-note p. 252).

□□□□□□□□□□$\mathscr{L}$□□□□□□□M_γ□□□□□□□γ□$\mathscr{L}$□□□□□□$\mathscr{K}$□□□□□□□□□□□$\mathscr{K}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathscr{L}_1$□□□□$\mathscr{L}_1$□□□□□W.F.F.□□□M□□□□□□□□□$\mathscr{L}_1$□□□□M□□□□□□□□□□□□□□□□$\mathscr{L}_1$□□□□□□□□□□□$\mathscr{L}_2$□$\mathscr{L}_2$□□□□□□M□□□□□□□□□□□□252□□□□□

□□□□□□□252□□5□□□□□□□□□□□□□□□□□□□□□□205□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□λ□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

The machine $\mathscr{L}$ includes $\mathscr{L}_2$ as a part. The motion of the machine $\mathscr{L}$ when supplied with the formula M_γ is divided into sections of which the n-th is devoted to finding the n-th figure of γ. The first stage in this n-th section is the formation of $\{M_\gamma\}(N_n)$. This formula is then supplied to the machine $\mathscr{L}_2$, which converts it successively into various other formulae. Each formula into which it is convertible eventually appears, and each, as it is found, is compared with

$$\lambda x\Big[\lambda x'\big[\{x\}\big(\{x\}(x')\big)\big]\Big], \quad i.e.\ N_2,$$

and with

$$\lambda x\big[\lambda x'[\{x\}(x')]\big], \quad i.e.\ N_1.$$

$\mathscr{L}_2$□□□□$\mathscr{L}$□□□□□□□□□□M_γ□□$\mathscr{L}$□□□□□□□□□□□□□□□□□□□□n□□□□□□γ□□n□□□n□□□□□□□□□□□$\{M_\gamma\}(N_n)$□□□□□□□□□□□□□$\mathscr{L}_2$□□

$$\lambda x\Big[\lambda x'\Big[\{x\}\big(\{x\}(x')\big)\Big]\Big], \text{ 即 } N_2,$$

□ $\lambda x\Big[\lambda x'\Big[\{x\}(x')\Big]\Big]$, 即 N_1。

□□□□□2□1□□□□□□□□□□□□□□□□□□□□□□λ□□□

If it is identical with the first of these, then the machine prints the figure 1 and the n-th section is finished. If it is identical with the second, then 0 is printed and the section is

finished. It it is different from both, then the work of $\mathscr{L}_2$ is resumed. By hypothesis, $\{M_\gamma\}(N_n)$ is convertible into one of the formulae N_2 or N_1; consequently the n-th section will eventually be finished, *i.e.* the n-th figure of γ will eventually be written down.

□□□□□□□□□□□□□□□□□□□□□1□□n□□□□□□□□□□□□□□□□□□□□□□0□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathscr{L}_2$□□□□□□□□□□□□□□□□□□□$\{M_\gamma\}(N_n)$□□□□□□□□□N_2□□N_1□□□□n□□□□□□□□□□□□□□□□□□γ□□n□□□□□□□□□

□□λ□□□□

To prove that every computable sequence γ is λ-definable, we must show how to find a formula M_γ such that, for all integers n,

$$\{M_\gamma\}(N_n) \text{ conv } N_{1+\phi_\gamma(n)}.$$

□□□□□□□□□□□□□□□γ□□λ□□□□□□□□□□□□□□□□□□□□□□□□□□□M_γ□□□□□□□□□n□

$$\left\{M_\gamma\right\}\left(N_n\right)\text{conv}N_{1+\phi_\gamma(n)}$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□N□□□□□□□□□□□□□□□□□□□□□□□□□□□□□λ□□□□□□□□□□□□□□□□□□□□□□□□λ□□□

Let $\mathscr{M}$ be a machine which computes γ and let us take some description of the complete configurations of $\mathscr{M}$ by means of numbers, *e.g.* we may take the D.N of the complete configuration as described in §6.

□□$\mathscr{M}$□□□□□□γ□□□□□□□□□□□□□□□□□□□$\mathscr{M}$□□□□□□□□□□§6□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□“□□□”□□□□□□□□□□□□□□□□□□□0, 1, 2, 3□□□m-□□□□□□□

Let $\xi(n)$ be the D.N of the n-th complete configuration of $\mathscr{M}$.

□$\xi(n)$□$\mathscr{M}$□□n□□□□□□□□□□□□□□

□□□□□□□□□□□□n□□□□□□□□□□□□□□$\xi(n)$□□□□□□□□□

The table for the machine $\mathscr{M}$ gives us a relation between $\xi(n+1)$ and $\xi(n)$ of the form

$$\xi(n+1) = \rho_\gamma\big(\xi(n)\big),$$

where ρ_γ is a function of very restricted, although not usually very simple, form: it is determined by the table for $\mathcal{M}$.

□$\mathcal{M}$□□□□□□□□□□□□□$\xi(n+1)$□$\xi(n)$□□□□□□□□□

$$\xi(n+1) = \rho_\gamma(\xi(n))$$

□□ρ_γ□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathcal{M}$□□□□□□□□

ρ_γ□□m-□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□m-□□□□□□□□

□□□□□□“□□□□□”□□10□□□□□□□□□□□□10□□□□□□□□□□□□□□□□ρ_γ□□□□□□□□□□□□□□□□□

ρ_γ is λ-definable (I omit the proof of this), *i.e.* there is a W.F.F.A_γ such that, for all integers n,

$$\{A_\gamma\}(N_{\xi(n)}) \text{ conv } N_{\xi(n+1)}.$$

ρ_γ□λ□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□A_γ□□□□□□□□□n□

$$\{A_\gamma\}(N_{\xi(n)}) \text{conv} N_{\xi(n+1)}$$

A_γ□□□□ρ_γ□□□□□□□□λ□□□□□□□□□□□□□□□□

Let U stand for

$$\lambda u\left[\{\{u\}(A_\gamma)\}(N_r)\right],$$

where $r = \xi(0)$;

□U□□

$$\lambda u\left[\{\{u\}(A_\gamma)\}(N_r)\right],$$

□□$r = \xi(0)$;

□□□□□U□□□□□γ□□□□□□□□□□□□□N_r□□□□□□□□□□□——□□313□□□□□□□□□DAD□□□m-□□q_1□DA□□□□□□D□□□u□□□□□□□□□□0□1□2□□□u□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

then, for all integers n,

$$\{U_\gamma\}(N_n) \text{ conv } N_{\xi(n)}.$$

□□□□□□□□□n□

$$\{U_\gamma\}(N_n)\,\mathrm{conv}\,N_{\xi(n)}$$

U_γ□□□□□□□□□0□1□2□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□4□□□□□□□□□□□□□□□□$\{U_\gamma\}(N_4)$□□□

$$\left\{\lambda u\left[\left\{\{u\}\left(A_\gamma\right)\right\}\left(N_{\xi(0)}\right)\right]\right\}\left(\lambda xy.x\left(x\left(x\left(x\left(y\right)\right)\right)\right)\right)$$

□U_γ□□□□□□□$N_{\xi(0)}$□□□N_r□□□□□□□□□□□□□□□□□□□□4□□□□□□□□

□u□

$$\left\{\left\{\lambda xy.x\left(x\left(x\left(x\left(y\right)\right)\right)\right)\right\}\left(A_\gamma\right)\right\}\left(N_{\xi(0)}\right)$$

□□□x□□□A_γ□

$$\left\{\lambda y.A_\gamma\left(A_\gamma\left(A_\gamma\left(A_\gamma\left(y\right)\right)\right)\right)\right\}\left(N_{\xi(0)}\right)$$

□□□□□$N_{\xi(0)}$□□y□

$$A_\gamma\left(A_\gamma\left(A_\gamma\left(A_\gamma\left(N_{\xi(0)}\right)\right)\right)\right)$$

A_γ□$N_{\xi(0)}$□□□□□□□□□$N_{\xi(1)}$□□□□□□$N_{\xi(2)}$□□□□□□□□□□□□$N_{\xi(4)}$□□□□□□□□□□□□□□□□□□□□□□u□□□U_γ□□□□□□□□□□□□□□□□□□□□A_γ□□□□u□□□□□

It may be proved that there is a formula V such that

$$\left\{\{V\}(N_{\xi(n+1)})\right\}(N_{\xi(n)})\begin{cases}\mathrm{conv}\,N_1 & \text{if, in going from the } n\text{-th to the } (n+1)\text{-th complete configuration, the figure 0 is printed.}\\ \mathrm{conv}\,N_2 & \text{if the figure 1 is printed.}\\ \mathrm{conv}\,N_3 & \text{otherwise.}\end{cases}$$

□□□□□□□□V□□

$$\left\{\{v\}\left(N_{\xi(n+1)}\right)\right\}\left(N_{\xi(n)}\right)\begin{cases}\text{conv}\,N_1 & \text{打印了0，从第}n\text{个完全格局转换至第}n+1\\ & \text{个完全格局，}\\ \text{conv}\,N_2 & \text{打印了1}\\ \text{conv}\,N_3 & \text{其他情况}\end{cases}$$

□□V□□□□□□□□□□□□□□□□□□□□0□□1□□□□□□□□□□□□□□□□□□
□□□□□

Let W_γ stand for

$$\lambda u\left[\{\{V\}\left(\{A_\gamma\}\left(\{U_\gamma\}(u)\right)\right)\}\left(\{U_\gamma\}(u)\right)\right],$$

so that, for each integer n,

$$\left\{\{V\}(N_{\xi(n+1)})\right\}(N_{\xi(n)})\ \mathbf{conv}\ \{W_\gamma\}\ (N_n),$$

□W_γ□□

$$\lambda u\left[\left\{\{V\}\left(\left\{A_\gamma\right\}\left(\left\{U_\gamma\right\}(u)\right)\right)\right\}\left(\left\{U_\gamma\right\}(u)\right)\right],$$

□□□□□□□□□□□n□

$$\left\{\{V\}\left(N_{\xi(n+1)}\right)\right\}\left(N_{\xi(n)}\right)\text{conv}\left\{W_\gamma\right\}\left(N_n\right),$$

□□□□□□□□□□□□□N_1□N_2□ N_3□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$\left\{W_\gamma\right\}\left(N_n\right)$$

□W_γ□□□□□□□□□□□□□□□

$$\left\{\lambda u\left[\left\{\{V\}\left(\left\{A_\gamma\right\}\left(\left\{U_\gamma\right\}(u)\right)\right)\right\}\left(\left\{U_\gamma\right\}(u)\right)\right]\right\}\left(N_n\right)$$

□N_n□□u□

$$\left\{\{V\}\left(\left\{\mathrm{A}_\gamma\right\}\left(\left\{\mathrm{U}_\gamma\right\}\left(N_n\right)\right)\right)\right\}\left(\left\{\mathrm{U}_\gamma\right\}\left(N_n\right)\right)$$

□□□$\{U_\gamma\}(N_n)$□□□□□$N_{\xi(n)}$, □□□

$$\left\{\{V\}\left(\left\{A_\gamma\right\}\left(N_{\xi(n)}\right)\right)\right\}\left(N_{\xi(n)}\right)$$

□□□$\{A_\gamma\}(N_{\xi(n)})$□□□□□$N_{\xi(n+1)}$□□□□□□□□□□

$$\left\{\{V\}\left(N_{\xi(n+1)}\right)\right\}\left(N_{\xi(n)}\right)$$

□□□□□□□□□□□□□□□□□$\{W_\gamma\}(N_n)$□□□□□N_1□N_2□ N_3□□□□□□n□□□□□□□□□$n+1$□□□□□□0□□1□□□□□□□□□□

and let Q be a formula such that

$$\left\{\{Q\}(W_\gamma)\right\}(N_s) \text{ conv } N_{r(z)},$$

where $r(s)$ is the s-th integer q for which $\{W_\gamma\}(N_q)$ is convertible into either N_1 or N_2.

□Q□□□□□□□□

$$\left\{\{Q\}\left(W_\gamma\right)\right\}\left(N_s\right)\text{conv}N_{r(z)} ,$$

□□$r(s)$□□s□□□q□□□$\{W_\gamma\}(N_q)$□□□N_1□N_2□

□□□□□□□□□□□□□M□□□□□□□□$r(s)$□□□$r(z)$□□□□□□□□□□□□□□□□0□□1□□□$r(s)$□□□□□□□□□□□□□□□□□□□□□1□4□6□7□□□□□□□□□0□□1□□□$r(1)$□□1□$r(2)$□□4□$r(3)$□□6□$r(4)$□□7□□□□

Then, if M_γ stands for

$$\lambda w\left[\{W_\gamma\}\left(\left\{\{Q\}(W_\gamma)\right\}(w)\right)\right],$$

it will have the required property†.

† In a complete proof of the λ-definability of computable sequences it would be best to modify this method by replacing the numerical description of the complete configurations by a description which can be handled more easily with our apparatus. Let us choose certain integers to represent the symbols and the m-configurations of the machine. Suppose that in a certain complete configuration the numbers representing the successive symbols on the tape are $s_1 s_2 \ldots s_n$, that the m-th symbol is scanned, and that the m-configuration has the number t; then we may represent this complete configuration by the formula

$$\left[[N_{s_1}, N_{s_2}, \ldots, N_{s_{m-1}}], [N_t, N_{s_m}], [N_{s_{m+1}}, \ldots, N_{s_n}]\right],$$

where

$$[a, b] \text{ stands for } \lambda u\left[\left\{\{u\}(a)\right\}(b)\right],$$

$$[a, b, c] \text{ stands for } \lambda u\left[\left\{\{\{u\}(a)\}(b)\right\}(c)\right],$$

etc.

□□M_γ□□

$$\lambda w\left[\{W_\gamma\}\left(\{\{Q\}(W_\gamma)\}(w)\right)\right],$$

□□□□□□□□□□□□□□□†

† □□□□□□□λ□□m-□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$S_1 S_{2\ldots} S_n$□□□□m□□□□□□□□□m-□□□□□□□□t□□□□□□□□□□□□□□□□□□□□□□□□

$$\left[\left[N_{s_1}, N_{s_2}, \cdots, N_{s_{m-1}}\right], \left[N_t, N_{s_m}\right], \left[N_{s_{m+1}}, \cdots, N_{s_n}\right]\right],$$

□□

$$[a,b] \text{ 代表 } \lambda u\left[\{\{u\}(a)\}(b)\right]$$

$$[a,b,c] \text{ 代表 } \lambda u\left[\{\{\{u\}(a)\}(b)\}(c)\right],$$

□□□

□□□□□□□□□□□□□□□□□□□□M_γ□□□□□□□n□

$$\{M_\gamma\}(N_n) \operatorname{conv} N_{1+\phi_\gamma(n)}$$

□□□□□□□□□□□□n□□0□□1□□□□□□□□

$$\{M_\gamma\}(N_n)$$

□□□□□□M_γ□□□□□□□□□□

$$\left\{\lambda w\left[\{W_\gamma\}\left(\{\{Q\}(W_\gamma)\}(w)\right)\right]\right\}(N_n)$$

□N_n□□w□

$$\{W_\gamma\}\left(\{\{Q\}(W_\gamma)\}(N_n)\right)$$

□□□□□□□□□□□□$N_{r(n)}$□□□

$$\{W_\gamma\}\left(N_{r(n)}\right)$$

[illegible] N_1 [illegible] N_2 [illegible] N_3 [illegible] $r(n)$ [illegible] 0 [illegible] 1 [illegible]
$r(n)$ [illegible] 0 [illegible] 1 [illegible]

[illegible] λ [illegible]

[illegible]

The Graduate College,
Princeton University,
New Jersey , U.S.A.

[illegible]

[illegible] “[illegible] λ [illegible]” [illegible] 1937 [illegible] 9 [illegible] 11 [illegible] [20] [illegible]

> [illegible] “[illegible]” [illegible] λ [illegible] λ [illegible]

[illegible] λ [illegible] λ [illegible]

[illegible] λ [illegible] · [illegible] λ [illegible] “λ [illegible]” [illegible]

[illegible] 1935 [illegible] [21] [illegible] λ [illegible]

---

[1] Maria Manzano□“Alonzo Church: His Life□His Work and Some of His Miracles”□*History and Philosophy of Logic*□Vol. 18□1997□□212□

[2] P. J. Landin□“A Correspondence Between ALGOL 60 and Church’s Lambda-Notation”□*Communications of the ACM*□Vol. 8□No.2□Feb 1965□□89-101□Vol. 8□No. 3□Mar 1965□□158-165□

[3] Herbert B. Enderton□“Alonzo Church: Life and Work”□introduction to *Collected Works of Alonzo Church*□MIT Press□forthcoming□□□□http:// www.math.ucla.edu/~hbe/church.pdf□

[4] Herbert B. Enderton, “In Memoriam: Alonzo Church, 1903–1995,” *The Bulletin of Symbolic Logic,* Vol. 1, No. 4 (1995), 486–488□

[5] □□□·□□□“A Set of Postulates for the Foundation of Logic”□*The Annals of Mathematics*□2nd Series□Vol. 33□No. 2□Apr.1932□□346-366□

[6] J. Barkley Rosser□“Highlights of the History of Lambda-Calculus”□*Annals of the History of Computing*□Vol. 6□No. 4□Oct.1984□□337-349□

[7] □□□“A Set of Postulates for the Foundation of Logic□Second Paper□”□*The Annals of Mathematics,* 2nd Series□Vol. 34□No. 4□Oct. 1933□□839-864□

[8] □□□842□

[9] □□□864□

[10] S. C. □□□□“Proof by Cases in Formal Logic”□*The Annals of Mathematics*□ 2nd Series□Vol. 35□No. 3□July 1934□□529-544□

[11] S. C. □□□□“A Theory of Positive Integers in Formal Logic□Part I”□*American Journal of Mathematics*□Vol. 57□No. 1□Jan. 1935□□153-173□Vol. 57□No. 2□Apr. 1935□□219-244□

[12] □□□□□□“Some Properties of Conversion”□*Transactions of the American Mathematical Society*□Vol. 39□No. 3□May 1936□□472-482□

[13] □□□□□□□□ “A Theory of Positive Integers in Formal Logic, Part I” □□10□□□□□□□

[14] □□□□“Origins of Recursive Function Theory”□*Annals of the History of Computing*□Vol. 3□No. 1□Jan 1981□□59□

[15] □□□□□□□□□□□□□□□□□□□□□□□□□□□□□1965□□□□□□□□□·□□□□ ed.□*The Undecidable*□Raven Press□1965□□41-71□□□□□□□□□□□□□□

·□□□□*Selected Works: Volume I, Publications* 1929-1936□Oxford University Press□1986□□346-371□

[16] □□□“An Unsolvable Problem of Elementary Number Theory”□*American Journal of Mathematics*□Vol. 58□No.2□Apr. 1936□□345-363□

[17] □□□□“General Recursive Functions of Natural Numbers”□*Mathematische Annalen*□Vol. 112□No.1□Dec.1936□□727-742□□□□□□·□□□□ed.□*The Undecidable*□Raven Press□1965□□237-252□S. C. □□□□“λ-Definability and Recursiveness”□*Duke Mathematical Journal*□Volume 2□Number 2□1936□□340-353□

[18] □□□“A Note on the Entscheidungsproblem”□*The Journal of Symbolic Logic*□Vol. 1□No. 1□Mar. 1936□□40-41□□□□□□□“Correction to a Note on the Entscheidungsproblem”□*The Journal of Symbolic Logic*□Vol. 1□No. 3□Sep. 1936□□101-102□

[19] □□□·□□□□*Alan Turing: The Enigma*□Simon & Schuster□1983□□116□

[20] □□·□□□“Computability and λ-Definability”□*The Journal of Symbolic Logic*□Vol. 2□No. 4□Dec. 1937□□pp.153-163□

[21] □□·□□□“Systems of Logic Based on Ordinals”□*Proceedings of the London Mathematical Society*□2nd Series□Vol. 45□No.1□1939□□161-228□

[illegible]16[illegible]

[illegible]

[illegible]

[illegible]”[illegible]”——19[illegible]·[illegible]1900—1979[illegible]”[illegible]”[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]1936[illegible]

[illegible][1][illegible]19[illegible]20[illegible]

[illegible]

[illegible]

[illegible]·[illegible]1908—1992[illegible]

> [illegible][2]

[illegible]·[illegible] “[illegible]”[3]

[illegible]

[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]·[illegible]

[illegible]100[illegible]

> [illegible]……[illegible][4]

[illegible]·[illegible]

[illegible]

“[illegible]”[illegible]384—[illegible]322[illegible]”[illegible]”[5]

[illegible]“[illegible]”[6]

[illegible]“[illegible]’[illegible]’[illegible]’[illegible]’[illegible]”[7]

[illegible]

> [illegible][8]

[illegible]

[illegible]·[illegible]1913—2000[illegible]

> [illegible][9]

[illegible]

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$$

$$\left(1+\frac{1}{\infty}\right)^{\infty}$$

n 2.71828…

∞

1815—1897

“ ” “ ” dx

19 · 1823—1891

“ ”[10] —— “ ”

——

·

[11]

1883

—— —— …… …… …… ……[12]

· · 20 · · · 1881—1966

20 “ ”

[illegible]1905[illegible]“[illegible]’[illegible]’[illegible]”[illegible][13]

[illegible]

[illegible]“[illegible]”[illegible]

——[illegible]

[illegible]·[illegible]

[illegible]1913[illegible]“[illegible]……[illegible]”[illegible][14][illegible]“[illegible]”[illegible][15]

1918[illegible]1928[illegible]

[illegible][16] [illegible]

[illegible]

$$C_n = (-\frac{1}{2})^n \quad n < k$$

$$C_n = (-\frac{1}{2})^k \quad n \geq k$$

[illegible] $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, -\frac{1}{32}$, [illegible] n [illegible] k [illegible]

[illegible] n [illegible] k [illegible] c_n [illegible] $\left(-\frac{1}{2}\right)^k$ [illegible]

[illegible] k [illegible]0123456789[illegible]π[illegible]

c_n [illegible] k [illegible] c_n [illegible]

[illegible] c [illegible]“[illegible]”[17]

[illegible]30[illegible][18][illegible]0123456789[illegible]π[illegible]17387594880[illegible]c_n[illegible]$2^{-17387594880}$[illegible]k[illegible]k[illegible]π[illegible]1[illegible]7[illegible]π[illegible]1[illegible]7[illegible]π[illegible]π[illegible]

[illegible]

[illegible]

$$-X \vee X$$

[illegible]

$$X \to X$$

[illegible]

$$-(X \,\&\, -X)$$

[illegible]$X \to X$[illegible]“[illegible]”[illegible]“[illegible]”[illegible][19]

[illegible]”[illegible]”[illegible]“[illegible]”[20][illegible]

[illegible]1928[illegible]*Mathematische Annalen*[illegible]85[illegible]

[illegible][21][illegible]30[illegible]5[illegible]30[illegible]

[illegible]“[illegible]”[illegible]real analysis[illegible]“[illegible]”[illegible]computable analysis[illegible][22]

[illegible]0[illegible]1[illegible]0[illegible]1[illegible]0123456789[illegible]π[illegible]0123456789[illegible]1[illegible]7[illegible]0[illegible]

[illegible]π[illegible]

[illegible]1921[illegible]“[illegible]”（Does Every Real Number Have a Decimal Expansion?）[23][illegible]π[illegible]

[illegible]1936[illegible]

1936[illegible]

[illegible]1927[illegible]

> [illegible]……[illegible][24]

“[illegible]”[illegible]80[illegible]1929[illegible]48[illegible]

[illegible]λ[illegible]“[illegible]”。[25]

[illegible]*Introduction to Metamathematics*（1952[illegible]*The Foundations of Intuitionistic Mathematics*（1965[illegible]1953[illegible]——[illegible]——[illegible][26]

[illegible]·[illegible]1930[illegible]1933[illegible]1919[illegible]1928[illegible]

[illegible]53[illegible]42[illegible]1936[illegible]11[illegible]20[illegible]1936[illegible]12[illegible]23[illegible]1936[illegible]10[illegible]1937[illegible]4[illegible]42[illegible]2[illegible]

[illegible]43[illegible]1937[illegible]12[illegible]30[illegible]43[illegible]2[illegible]1937[illegible]5[illegible]12[illegible]

[544]

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM. A CORRECTION

By A. M. TURING.

In a paper entitled “On computable numbers, with an application to the Entscheidungsproblem”[*] the author gave a proof of the insolubility of the Entscheidungsproblem of the “engere Funktionenkalkül”. This proof contained some formal errors[†] which will be corrected here: there are also some other statements is the same paper which should be modified, although they are not actually false as they stand.

---

[*] *Proc. London Math. Soc.* (2), 42(1936-7), 230-265.

[†] The author is indebted to P. Bernays for pointing out these errors.

[illegible]

A. M. [illegible]

[illegible][*][illegible]“[illegible]”[illegible]engere Funktionenkalkül[illegible][†][illegible]

---

[*] [illegible](2) 42 (1936-7), 230-265[illegible]

[†] [illegible]

[illegible]3[illegible]11[illegible]14[illegible]

The expression for Inst $\{q_i S_j S_k L q_l\}$ on p.260 of the paper quoted should read

$$(x,y,x',y')\Big\{\Big(R_{S_j}(x,y)\ \&\ I(x,y)\ \&\ K_{q_i}(x)\ \&\ F(x,x')\ \&\ F(y',y)\Big)$$
$$\to\Big(I(x',y')\ \&\ R_{S_k}(x',y)\ \&\ K_{q_l}(x')\ \&\ F(y',z)\ \mathrm{v}\ \Big[\big(R_{S_0}(x,z)\to R_{S_0}(x',z)\big)$$
$$\&\ \big(R_{S_1}(x,z)\to R_{S_1}(x',z)\big)\ \&\ \dots\ \&\ \big(R_{S_M}(x,z)\to R_{S_M}(x',z)\big)\Big]\Big)\Big\},$$

$S_0, S_1, \dots, S_M$ being the symbols which $\mathscr{M}$ can print.

[illegible]260[illegible] Inst$\{q_i S_j S_k L q_l\}$[illegible]

$$(x,y,x',y')\Big\{\Big(R_{S_j}(x,y)\ \&\ I(x,y)\ \&\ K_{q_i}(x)\ \&\ F(x,x')\ \&\ F(y',y)\Big)$$

$$\rightarrow\Big(I(x',y')\,\&\,R_{s_k}(x',y)\,\&\,K_{q_l}(x')\,\&\,F(y',z)\,\mathrm{v}\Big[\big(R_{S_0}(x,z)\rightarrow R_{S_0}(x',z)\big)$$

$$\&\big(R_{S_1}(x,z)\rightarrow R_{S_1}(x',z)\big)\,\&\ldots\&\,\big(R_{S_M}(x,z)\rightarrow Rs_M(x',z)\big)\Big]\Big)\Big\},$$

S_0, S_1,..., S_M[illegible]$\mathcal{M}$[illegible]

[illegible]$F(y',z$[illegible]z[illegible]

[illegible]250[illegible]

The statement on p. 261, line 33, viz.

$$\text{“Inst}\{q_a\, S_b\, S_d\, L q_c\}\ \&\ F^{(n+1)} \rightarrow (CC_n \rightarrow CC_{n+1})$$

is provable″ is false (even with the new expression for Inst $\{q_aS_bS_dLq_c\}$): we are unable for example to deduce $F^{(n+1)}\rightarrow(-F(u,u''))$ and therefore can never use the term

$$F(y',z)\ \mathrm{v}\ \Big[\big(R_{S_0}(x,z)\rightarrow R_{S_0}(x',z)\big)\,\&\ldots\&\,\big(R_{S_M}(x,z)\rightarrow R_{S_M}(x',z)\big)\Big]$$

[545]

in Inst $\{q_aS_bS_dLq_c\}$.

[illegible]261[illegible]33[illegible]

$$\text{“Inst}\{q_aS_bS_dLq_c\}\ \&\ F^{(n+1)} \rightarrow \big(CC_n \rightarrow CC_{n+1}\big)$$

[illegible]”[illegible]Inst$\{q_aS_bS_dLq_c\}$[illegible]$F^{(n+1)}\rightarrow(-F(u,u''))$[illegible]Inst$\{q_aS_bS_dLq_c\}$[illegible]

$$F(y',z)\,\mathrm{v}\Big[\big(R_{S_0}(x,z)\rightarrow R_{S_0}(x',z)\big)\,\&\ldots\&\,\big(R_{S_M}(x,z)\rightarrow R_{S_M}(x',z)\big)\Big]$$

[illegible]

[illegible]

To correct this we introduce a new functional variable G [$G(x, y)$ to have the interpretation “x precedes y″]. Then, if Q is an abbreviation for

$$(x)(\exists w)(y,z)\Big\{F(x,w)\ \&\ \big(F(x,y)\rightarrow G(x,y)\big)\ \&\ \big(F(x,z)\ \&\ G(z,y)\rightarrow G(x,y)\big)$$

$$\&\ \Big[G(z,x)\ \mathrm{v}\ \big(G(x,y)\ \&\ F(y,z)\big)\ \mathrm{v}\ \big(F(x,y)\ \&\ F(z,y)\big)\rightarrow\big(-F(x,z)\big)\Big]\Big\}$$

the corrected formula Un($\mathcal{M}$) is to be

$$(\exists u)A(\mathcal{M}) \to (\exists s)(\exists t)\, R_{S_1}(s,t),$$

where A($\mathcal{M}$) is an abbreviation for

$$Q \;\&\; (y)R_{S_0}(u,y) \;\&\; I(u,u) \;\&\; K_{q_1}(u) \;\&\; \mathrm{Des}(\mathcal{M}).$$

The statement on page 261 (line 33) must then read

$$\mathrm{Inst}\{q_a\, S_b\, S_d\, Lq_c\} \;\&\; Q \;\&\; F^{(n+1)} \to (CC_n \to CC_{n+1}),$$

and line 29 should read

$$r\big(n,\, i(n)\big) = b, \quad r\big(n+1,\, i(n)\big) = d, \quad k(n) = a, \quad k(n+1) = c.$$

For the words “logical sum” on p.260, line 15, read “conjunction”. With these modifications the proof is correct. Un($\mathcal{M}$) may be put in the form (I)(p.263) with n=4.

[illegible]G [G(x[illegible]y), [illegible]“x[illegible]y[illegible]”][illegible]Q[illegible]

$$(x)(\exists w)(y,z)\Big\{F(x,w) \,\&\, \big(F(x,y) \to G(x,y)\big) \,\&\, \big(F(x,z) \,\&\, G(z,y) \to G(x,y)\big)$$
$$\&\Big[G(z,x) \vee \big(G(x,y) \,\&\, F(y,z)\big) \vee \big(F(x,y) \,\&\, F(z,y)\big) \to \big(-F(x,z)\big)\Big]\Big\}$$

[illegible]Un($\mathcal{M}$)[illegible]

$$(\exists u)\, A(\mathcal{M}) \to (\exists s)(\exists t)\, R_{S_1}(s,t)\ ,$$

[illegible]A($\mathcal{M}$)[illegible]

$$Q \,\&\, (y)\, R_{S_o}(u,y) \,\&\, I(u,u) \,\&\, Kq_1\,(u) \,\&\, \mathrm{Des}\,(\mathcal{M}).$$

[illegible]261[illegible]33[illegible]

$$\mathrm{Inst}\{q_a S_b S_d L q_c\} \,\&\, Q \,\&\, F^{(n+1)} \to \left(CC_n \to CC_{n+1}\right)\ ,$$

[illegible]29[illegible]

$$r\big(n, i(n)\big) = b, \quad r\big(n+1, i(n)\big) = d,\ \ k(n) = a, \quad k(n+1) = c\,.$$

[illegible]260[illegible]“[illegible]”[illegible]“[illegible]”[illegible]Un($\mathcal{M}$) [illegible] n=4[illegible]263[illegible]I[illegible]

[illegible]

Some difficulty arises from the particular manner in which “computable number” was defined (p.233).

[illegible]“[illegible]”[illegible]233[illegible]

[illegible]68[illegible]“[illegible]”

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

If the computable numbers are to satisfy intuitive requirements we should have:

If we can give a rule which associates with each positive integer n two rationals a_n, b_n satisfying $a_n \leq a_{n+1} < b_{n+1} \leq b_n$, $b_n - a_n < 2^{-n}$, then there is a computable number *α for which $a_n \leq a \leq b_n$ each n.*

(A)

[illegible]

[illegible]n[illegible]a_n[illegible]b_n[illegible]$a_n \leq a_{n+1} < b_{n+1} \leq b_n, b_n - a_n < 2^{-n}$, [illegible]$n$[illegible]$\alpha$[illegible]

$$a_n \leq a \leq b_n$$[illegible]

(A)

[illegible]“[illegible]”[illegible]0[illegible]n[illegible]2^{-n}[illegible]1, 0.1, 0.01, 0.001, [illegible]——[illegible]n[illegible]

A proof of this may be given, valid by ordinary mathematical standards, but involving an application of the principle of excluded middle.

[illegible]

[illegible]π[illegible]Turingesque[illegible]

On the other hand the following is false:

There is a rule whereby, given the rule of formation of the sequences a_n, b_n in (A) *we can obtain a* D.N. *for a machine to compute α.*

(B)

[illegible]

[illegible](A)[illegible]a_n[illegible]b_n[illegible]α[illegible]

(B)

[illegible]

That (B) is false, at least if we adopt the convention that the decimals of numbers of the form $m/2^n$ shall always terminate with zeros, zeros, can be seen in this way.

[illegible]$m/2^n$[illegible](B)[illegible]

□□$m/2^n$□□□□□□□□□□□□□□□□□□□□□□□□□1□□□□
□□□12345/65536□□□□□□□□0.0011 0000 0011 1001 0000…□□□
65536□2^{16}□□□□□□□□□□16□□□□□□□□□0□□□□□□□□□□□□□□0□□□m□□2^n
□□□$m/2^n$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□a_n□b_n□□□□□□□□□□□□□□□□□□□□□□$\mathcal{N}$□□□□□□□□□□

Let $\mathcal{N}$ be some machine, and define c_n as follows: $c_n = \frac{1}{2}$ if $\mathcal{N}$ has not printed a figure 0 by the time the n-th complete configuration is reached $c_n = \frac{1}{2} - 2^{-m-3}$ if 0 had first been printed at the m-th

[546]

complete configuration ($m \leq n$). Put $a_n = c_n - 2^{-n-2}$, $b_n = c_n + 2^{-n-2}$.

□$\mathcal{N}$□□□□□□□□□□c_n□□□□□□n□□□□□□$\mathcal{N}$□□□□□□□□□0□□$c_n = \frac{1}{2}$□□□□□ m□$m \leq n$□□□□□□□□□□□□□□0□□$c_n = \frac{1}{2} - 2^{-m-3}$□□□$a_n = c_n - 2^{-n-2}$, $b_n = c_n + 2^{-n-2}$□

□□□□□□□□□□□$\mathcal{N}$□□□□□□□□1111101…□□□□□□□c_n□a_n□b_n□□□□□□

n	序列	c_n	a_n	b_n
0	1	$\frac{1}{2}$	$\frac{1}{2}-\frac{1}{4}=\frac{64}{256}$	$\frac{1}{2}+\frac{1}{4}=\frac{192}{256}$
1	1	$\frac{1}{2}$	$\frac{1}{2}-\frac{1}{8}=\frac{96}{256}$	$\frac{1}{2}+\frac{1}{8}=\frac{160}{256}$
2	1	$\frac{1}{2}$	$\frac{1}{2}-\frac{1}{16}=\frac{112}{256}$	$\frac{1}{2}+\frac{1}{16}=\frac{144}{256}$
3	1	$\frac{1}{2}$	$\frac{1}{2}-\frac{1}{32}=\frac{120}{256}$	$\frac{1}{2}+\frac{1}{32}=\frac{136}{256}$
4	1	$\frac{1}{2}$	$\frac{1}{2}-\frac{1}{64}=\frac{124}{256}$	$\frac{1}{2}+\frac{1}{64}=\frac{132}{256}$
5	0	$\frac{127}{256}$	$\frac{127}{256}-\frac{1}{128}=\frac{125}{256}$	$\frac{127}{256}+\frac{1}{128}=\frac{129}{256}$
6 …	1	$\frac{127}{256}$	$\frac{127}{256}-\frac{1}{256}=\frac{126}{256}$	$\frac{127}{256}+\frac{1}{256}=\frac{128}{256}$

□□□□0□□□□□□□□□c_n□□$\frac{1}{2}$□□□□□□0□□□□□□□□□□m□□□□□5□□□□□n□□□□□m□□□□□c_n□□□$(2^{(m+2)}-1)/2^{(m+3)}$□

Then the inequalities of (A) are satisfied,

□□(A)□□□□□□□□

a_n□□□□□□□□b_n□□□□□□□□a_n□b_n□□□□□□□□□2^{-n}□□□□□□□□□□2^{-n-1}□□

and the first figure of α is 0 if $\mathcal{N}$ ever prints 0 and is 1 otherwise.

□□$\mathcal{N}$□□□0□□□α□□□□□□□0□□□$\mathcal{N}$□□□□□0□□□α□□□□□□□□1□

□□□□□□□□□□□□□127/256□□□□□□□□□□α□□□011111110000…□□□□□□□□□□□□n□□4□□□□□□□□□□□0□□□□□□63/128□α□□□□011111100000…□□□0□□□□□□□□□□□□□□□□$\frac{1}{2}$□□□□□□□□100000000000…□

□□□□□□□□□□a_n□b_n□□□□□□□□□□□□□□□□□□$\mathcal{N}$□□□□□□□□□□□□□□□□□□□□□□□□□□$\mathcal{N}$□□□□□□□□□□□0□

□□$\mathcal{N}$□□□□0□□□α□□□□□□□1□□□$\mathcal{N}$□□□0□□□α□□□□□□□0□□□□

If (B) were true we should have a means of finding the first figure of α given the D.N. of $\mathcal{N}$: *i.e.* we should be able to determine whether $\mathcal{N}$ ever prints 0, contrary to the results of §8 of the paper quoted.

□□(B)□□□□□□□$\mathcal{N}$□ D.N.□□□□□□□□□□□□α□□□□□□□□□□□□□□□□□$\mathcal{N}$□□□□□0□□□□□□□□□§8□□□□□□□□

□□□“□□□□”□□□

Thus although (A) shows that there must be machines which compute the Euler constant (for example) we cannot at present describe any such machine, for we do not yet know whether the Euler constant is of the form $m/2^n$.

□□□□□(A)□□□$m/2^n$□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□e□□□□□□□□□□□□□□γ□gamma□□□□□□□□□

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n}-\ln(n)\right)$$

□□□□□□□□□□□□□□□□□□□□□l/x□□□□□□□□□□

$$\gamma = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{i}-\int_{1}^{n}\frac{1}{x}\,\mathrm{d}x\right)$$

□□□□□□0.57721566490153286…□

□□□□□□□□□□□□–□□□□□□□□□□□1734□□□□□□□□□□□□□□□□□1781□□□□□□□□□□□16□□□□□□□1750—1800□□□□□□□□□□□□□□□□□□□□□1790□□□□□□□□32□□□□□□□19□□□□□□□□□□□□□□□□γ□□□□□□□□□[27]

□□□1937□□□□□□□□□□2008□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□“□□□□”□□□□□□□□□□□□□□□□□□□□“□□□□□□□□□□□□□□□□□□□□□□□□□□C□□□□□□□□□□□□□□□□□□□□□□2^n+1□□□□□□□□□□□□□□□□□□□□□□□……”□□—□□23□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□2□□□□□□□□□□□□□□□□□□□□□□□□□□□□0□□□

□□□□□□□□□□□□γ□□□□□□□□□□□□□□□□□γ□□□□$m/2^n$□□□□□□□□□□□□□□□□□□□□□□□□□□1□□□□□□□□□□□□□□□

[illegible]“[illegible]”γ[illegible]0[illegible]γ[illegible]

[illegible]$m/2^n$[illegible]m[illegible]2^n[illegible]n[illegible]0[illegible]$m/2^n$[illegible]n[illegible]0[illegible]0[illegible]

[illegible]

This disagreeable situation can be avoided by modifying the manner in which computable numbers are associated with computable sequences, the totality of computable numbers being left unaltered. It may be done in many ways* of which this is an example.

* This use of overlapping intervals for the definition of real numbers is dud originally to Brouwer.

[illegible]*[illegible]

* [illegible]

[illegible]

[illegible]

[illegible]

[illegible]“[illegible]”[illegible]

[3,4]

[3.1,3.2]

[3.14,3.15]

[3.141,3.142]

…

[illegible]π[illegible]“[illegible]”[illegible]π[illegible]halo[28][illegible]halo[illegible]halo[illegible]π[illegible]

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□……□□□□□□□□□□□□
□□
□□□□[29]

□□□□□□□□□□□□□□□□□□□——□□□□□□□□□□□□□□□□□□□0□□□□

□□□□□□□□□□□□□□□□□□□□□□□□□“□□□”□lawlike□□□□□□□□□“□□□
□”□□
□□□□□□

□□□
□□□□□□□□□□□subject□□□□□□□transcendent ego□□□□□□□
□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□“□□□□□□□”□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□[30]

□□
□□□

$$[3,4] \qquad [3,4]$$

$$[3.1,3.2] \qquad [3.1,3.2]$$

$$[3.14,3.15] \qquad [3.14,3.15]$$

$$[3.141,3.142] \qquad [3.141,3.142]$$

□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$[3.141,3.14175] \qquad [3.14125,3.142]$$

□□□□□□□□□□□□□□□□

$$[3.14125,3.14175] \qquad [3.14125,3.14175]$$

□□□□□□□□□□□□□□□□
□□□

□□□
□□
□□
□□□
□“□□□”□□□“□□□□”□□□□□□□□□□□□□□□[31]

[illegible]π/4[illegible]

11010010000111111011010101000100 0...

[illegible]π/4[illegible]/[illegible]π/4[illegible]0[illegible]1[illegible]π/4[illegible]1[illegible]0.1[illegible]0.10000000...[illegible]0.11111111...[illegible]0.1[illegible]0.11111111...[illegible]1[illegible]

[1.1,1.0]

[0.11,1.00]

[0.110,0.111]

[0.1100,0.1101]

[0.11001,0.11010]

[0.110010,0.110011]

[0.1100100,0.1100110]

...

[illegible]

[illegible]21[illegible]1928[illegible]*Die Struktur des Kontinuums*[illegible][32][illegible]0[illegible]1[illegible]0[illegible]1[illegible]

[illegible]

Suppose that the first figure of a computable sequence γ is i and that this is followed by 1 repeated n

times, then by 0 and finally by the sequence whose r-th figure is c_r; then the sequence γ is to correspond to the real number

$$(2i-1)n+\sum_{r=1}^{\infty}(2c_r-1)\left(\tfrac{2}{3}\right)^r.$$

[illegible]γ[illegible]i[illegible]n[illegible]1[illegible]0[illegible]r[illegible]c_r[illegible]γ[illegible]

$$(2i-1)n+\sum_{r=1}^{\infty}(2c_r-1)\left(\frac{2}{3}\right)^r$$

□□i□□□□□□□□0□□□□□1□□□□□n□□□□1□□□□□□□□□□0□□□□□□□□□□□□□□
□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

$$\pm\frac{2}{3}\ \pm\ \frac{4}{9}\ \pm\ \frac{8}{27}\ \pm\ \frac{16}{81}\ \pm\ \frac{32}{243}\ \pm\ \cdots$$

□□c_r□□□□□□□□□□□1□□□□□□□□□□0□□□

□□□□□□□□□□□□□□□□□

1 111110 11011…

□□□□□□□□□□□□□□□□□□□□□□□□□1□□□□□□□□□□□□□5□□□□□1□□0□□□□□
□□□□□□5□□□□□□

$$\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\frac{16}{81}+\frac{32}{243}$$

□□□□□□□□□□□□□□$5\frac{278}{243}$□$6\frac{35}{243}$□□□□□□□□□□□□□□□□□□□6□□□□5□□□□
□□□□□□□□□□1□□□□□□□□□□□

$$\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\frac{16}{81}+\frac{32}{243}+\cdots$$

□□□□□2□□□□□□□□□□□□□□□□□0□□□□□□□□□□□□2□□□□□□□□□□□□N□□□□
□□□□□□N-2□N+2□□□□□□□□□□□□□□□

If the mechine which computes γ is regarded as computing also this real number then □B) holds.

□□□□γ□□□□□□□□□□□□□□□□□□□(B)□□□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□a_n□□b_n□□□□□□□□□□□□□□□□□□□
□□□□□a_n□b_n□□□□□□□□□□□$(2/3)^n$□□□□□□□□□□□□□□□□□□□□□□0□□1□

□□$\frac{1}{2}$□

$\frac{1}{4}$□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□$\frac{1}{2}$□□□□□□□

001010 1101 0100 1101…

□□□□□□□□□□□□□□□□□□□□□□□2/3, -/4/9, 8/27, 416/81□□□□□□□□□□□
□□□□□□3□□□□□□10□□□□□□□□□□□□$\frac{1}{2}$□

□□□□□□□□□□□□□□□□□□□□□□□□□□□□$m/2^n$□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□□□0□□□□□□□□□0□1□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□□

[illegible]$\frac{1}{2}$[illegible]

0 10 0101 0010 1011 0010…

[illegible]1[illegible]-2/3, 4/9, -/8/27, 16/81[illegible]

The uniqueness of representation of real numbers by sequences of figures is now lost, but this is of little theoretical importance, since the D.N.'s are not unique in any case

The Graduate College,
Princeton, N.J., U.S.A.

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[1] □□Hal Hellman□*Great Feuds in Mathematics: Ten of the Liveliest Disputes Ever*□Wiley□2006□□□□□□□□□□□

[2] □□□·□□□□Mathematics: The Loss of Certainty□Oxford University Press□1980□□323□

[3] □□·□□□□The Road to Reality: A Complete Guide to the Laws of the Universe□Alfred A. Knopf□2005□□13□

[4] □□Ben H. Yandell□The Honors Class□Hilbert's Problems and Their Solvers□A. K. Peters□2002□□395□

[5] □□□□□□Physics□□□Robin Waterfield□Oxford World's Classics□1996□□Book III□□4□□65□

[6] □□□Book III□□6□□72□

[7] □□□Book III□□6□□73□

[8] □□□Book III□□8□□76-7□

[9] □□□·□□□□"Continuity"in Paul Edwards□ed.□*The Encyclopedia of Philosophy*□Macmillan□1967□□Vol. 2□205□

[10] □□□□□□□□□□□□□□□□□□□□□□□□□□□1893□□"Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk"□□□William Ewald□ed.□*From Kant to Hilbert□A Source Book in the Foundations of Mathematics*□Oxford University Press□1996□□Vol. II□942□□1922□□"The New Grounding of Mathematics□ First Report"□□ □□□□□□□□□□□□□□□□□"Die ganze Zahl schuf der liebe Gott, alles andere ist Menschenwerk"□□*From Kant to Hilbert□*Vol. II□1120□

[11] Harold Edwards□"Kronecker's Place in History"□in William Aspray and Philip Kitcher□eds.□*History and Philosophy of Modern Mathematics*□University of Minnesota Press□1988□□139-144□

[12] □□□□·□□□□"Foundations of a General Theory of Manifolds□A Mathematico-Philosophical Investigation into the Theory of the Infinite"□in *From Kant to Hilbert□*Vol. II□ pgs.895-896□

[13] □□□□"Brower's Intuitionist Program"□in Paolo Mancosu□ed.□*From Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s*□Oxford University Press□1998□□5□

[14] □□□□□"Intuitionism and Formalism"□*Bulletin of the American Mathematical Society□*Vol. 20□1913□□81-96□

[15] □□□

[16] □□□□□"On the Significance of the Principle of Excluded Middle in Mathematics, Especially in Function Theory"□1923□□in jean van Heijenoort. ed.□*From Frege to Gödel□A Source Book in*

Mathematical Logic□1879-1931□Harvard University Press□1967□□337□

[17] □□□□·□□□*Hilbert*□Springer□1970□1996□□184□

[18] Jonathan Borwein□The Brouwer-Heyting Sequence□http://www.cecm.sfu.ca/~jborwein/brouwer.html□

[19] F1oy E. Andrews□“The Principle of Excluded Middle Then and Now: Aristotle and *Principia Mathematica”*□*Animus*□Vol. 1□1996□□http://www2.swgc.mun.ca/animus/1996vol1/andrews.pdf□

[20] □□□□□“On the Infinite”□1925□□□□*From Frege to Gödel*□□□375-376□

[21] G. Kreisel □□□□“Luitzen Egbertus Jan Brouwer.1881-1966”□*Biographical Memoirs of Fellows of the Royal Society*□Vol. 15□Nov. 1969□□39-68□

[22] □□□□□Oliver Aberth□*Computable Analysis*□McGraw-Hill□1980□□

[23] □*From Brouwer to Hilbert*□□□□□28-35□

[24] William Aspray□The Princeton Mathematics Community in the 1930s: An Oral-History Project□An interview with Alonoz Church at the University of California On 17 May 1984□http://www.princeton. edu/~mudd/finding_aids/mathoral/pmco5.htm□

[25] □□□“A Set of Postulates for the Foundation of Logic”□*The Annals of Mathematics*□second Series□Vol. 33□No. 2□Apr. 1932□□346□

[26] □□□□“Origins of Recursive Function Theory□” *Annals of the History of Computing,* Vol. 3. No. 1□Jan. 1981□□62.

[27] Julian Havil□*Gamma*□*Exploring Euler’s Constant*□Princeton University Press□2003□□89□

[28] □□□□□□Edmund Husserl□1859——1938□□□□ Mark van Atten□Dirk van Dalen□and Richard Tieszen□“Brouwer and Weyl: The Phenomenology and Mathematics of the Intuitive Continuum”□Philosophia Mathematica□Vol. 10□No. 2□2002□□207□

[29] Mark van Atten□*On Brouwer*□Wadsworth□2004□□31□

[30] van Atten□van Dalen□and Tieszen□“Brouwer and Weyl: The Phenomenology and Mathematics of the Intuitive Continuum□”212□

[31] van Atten□van Dalen□and Tieszen□“Brouwer and Weyl: The Phenomenology and Mathematics of the Intuitive Continuum□”212-213□

[32] □□□□□“The Structure of the Continuum”□*From Kant to Hilbert*□Vol. II□1186-1197□

[illegible]17[illegible]

[illegible]

[illegible][1]

[illegible]20[illegible]“[illegible]”[illegible]

[illegible]0[illegible]1[illegible]——[illegible]——[illegible]

[illegible]10[illegible]219[illegible]0[illegible]1[illegible]0[illegible]1[illegible]0[illegible]0[illegible]0[illegible]1[illegible]1[illegible]

[illegible]·[illegible]·[illegible]1934[illegible]

[illegible]“[illegible]$ø(x)$[illegible]”[2][illegible]

1941[illegible]1952[illegible]8[illegible]

[illegible]tick[illegible]tally[illegible]tick[illegible]0[illegible]tick[illegible]0[illegible]2[illegible]tick[illegible]1[illegible][3]

[illegible]tick[illegible]

表这个元素在过程执行的那一刻。让过程以自动机的方式操作，提供所需的外部条件，将按照一个tick那样转变，每一tick都将是程序提供的结构性。

大致的想法是一种运算的自动机，就像现在通常被描述的那样，但是人们可以想象，自动机可以表现出类似的行为，比如，"操作"和"物质"。冯·诺伊曼明白，他说的"物质器官"并不像它与电子计算机的可能性那样简单，但是它们的有效性也是要回应，他感受到，正如他强调的"我们要把神经系统理解为一种计算机器，但从前提来说这不能说是一种数字机器"。[4]

有一个有点具有讽刺意义的部分——冯·诺伊曼既有的不同部分说明了是有数学精确性的计算机是如何一个小步骤操作起来，从这个角度也可以知道，一台计算机可以每次都做许多小步骤，所以我们必须把"方案"的执行看作成功，而不是一种人的工作。但是这种自动机是有关联的，然而冯·诺伊曼在论文中没有提到。

图灵在15年前即已提出了1936年的论文，在3个方面提出了这个机器的设想中的逻辑问题：

〉判定问题

〉1934年库尔特·哥德尔提出关于计算函数的理论，以及判定问题的不可解答的问题。

〉阿隆佐·邱奇在同一个时期的研究中发展的λ演算的可计算性。

图灵关于这个判定问题的论文中的设计描述在1936年发表后，邱奇写了一篇关于图灵论文的评论文章发表在1937年，描述"图灵机与λ可定义性"的等价性。阿隆佐·邱奇在1936年发表了"λ可定义的函数理论"，他是第一个说出了"可计算的"是"有效可计算的"这个事实的人。

库尔特·哥德尔在一篇演讲中说道，所有能做到数学计算的机器能够在有效计算的领域里面做什么，他认为图灵的机器是一个"有效可计算函数的正确定义，而不仅仅是一个关于它的等效的定义"，因为图灵在论文中对"有效可计算的函数就是可以由某种机器所计算的函数，也就是说，它可以作为一个有限的机械程序来执行……"[5]

邱奇1967年的一次访谈中说到，他是这样描述的：

图灵的概念是一台计算机器和它所能做的所有事情，从"机器的运算"的概念"有效可计算的"的概念角度来看，把它看作一个"可计算"的过程。[6]

丘奇说的是"人类的计算者"在现代的计算机里所做的。

冯·诺伊曼在普林斯顿高等研究院完成了6年的工作，他有一台计算机器在那里，也可能是世界上最早的存储程序计算机之一。

艾伦·图灵在1928年开始了研究工作，1950年他在论文中提到了人工智能的概念，这是一个机器是否能思考的问题，以及与图灵在1934年开始，直到1938年完成的工作。

他的论文中描述了一台机器可以像人类一样去计算或者"模仿游戏"，这篇论文是在1952年的[7]。冯·诺伊曼在1958年的一本书中提到了一个与人的大脑有关的计算机的比较方面，他在书中写道："计算机可以看作是从事计算的机器，而人的大脑也可以在某种程度上看作是一台计算机器"，他还在书中说明："机器是不能自我复制的，除非它可以被设计成那样。"[8]

[illegible]321[illegible]13[illegible]1[illegible]

[illegible]tick[illegible]12[illegible]

[illegible]“[illegible]”[illegible][9]

[illegible]McGraw-Hill[illegible]”[illegible]“[illegible]“[illegible]”[illegible]1958[illegible]1959[illegible]

[illegible]·[illegible]

[illegible]70[illegible]·[illegible]

> [illegible]Z[illegible]Z[illegible]
>
> [illegible]α[illegible]α[illegible]Z[illegible]
>
> [illegible]Z[illegible]Z[illegible][10][illegible]

[illegible]70[illegible]“[illegible]”

[illegible]

[illegible]

[illegible]Z3[illegible]·[illegible]1938[illegible]1941[illegible][11][illegible]·[illegible]19[illegible]30[illegible]1944[illegible]

[illegible]35mm[illegible]

[illegible]

[illegible]

[illegible]HTML[illegible]HTML[illegible]JavaScript[illegible]

[illegible]

[illegible]“[illegible]”[illegible]

[illegible]

[illegible]1939[illegible]*Systems of Logic Based on Ordinals*[illegible]“[illegible]”[illegible]

> [illegible]o-[illegible][12]

[illegible]1/2[illegible]·[illegible][13]

[illegible]

[illegible]

[illegible]“[illegible]”[illegible]

[illegible][14]

[illegible]

[illegible]1936[illegible]·[illegible]1898—1969[illegible]·[illegible]1923—1969[illegible]

[illegible]·[illegible]15[illegible]·[illegible]·[illegible]

[illegible]1891—1970[illegible]1937[illegible]·[illegible]·[illegible][15]

[illegible]·[illegible][16]

[illegible]12[illegible]3[illegible][17]

[illegible]1938[illegible][18]

[illegible]“[illegible]”[illegible][19][illegible]4F[illegible][20][illegible]“[illegible]”[illegible][21][illegible]·[illegible]1899—1972[illegible]

[illegible]1924[illegible]1934[illegible]“[illegible]”[illegible]1935[illegible]1938[illegible]1939[illegible][22]

1942[illegible]1943[illegible]3[illegible]·[illegible]

[illegible]·[illegible]“[illegible]”[illegible]1917[illegible]·[illegible]1863—1948[illegible]“[illegible]……[illegible]?”[illegible]“[illegible]”[illegible]·[illegible]“[illegible]”[23]

[illegible]1927[illegible][24]1934[illegible]·[illegible]·[illegible]1885—1940[illegible]1941[illegible]

[illegible]“[illegible]”[25][illegible]“[illegible]”，[26][illegible]·[illegible]·[illegible]

[illegible]·[illegible]·[illegible]1943[illegible][27]

[illegible]19[illegible]20[illegible][28]

[illegible]

[illegible]19[illegible]15[illegible]

> [illegible]……[illegible]λ[illegible][29]

[illegible]1948[illegible]

> [illegible]……[illegible][30]

[illegible] 1955 [illegible] “[illegible]”[31] [illegible]

> [illegible] [32]

[illegible] 20 [illegible]·[illegible]·[illegible]

[illegible]·[illegible]·[illegible]·[illegible]

11 [illegible] 14 [illegible] 18 [illegible] “[illegible]” [illegible] “[illegible]”[33] [illegible] 15 [illegible]

[illegible]·[illegible] 1919 [illegible]

[illegible]·[illegible] MIT [illegible]·[illegible]

[illegible] 1942 [illegible] 5 [illegible] 13 [illegible]·[illegible]·[illegible]·[illegible] “[illegible]” [illegible][34] [illegible]·[illegible]

1947 [illegible] cybernetics [illegible] “[illegible]” [illegible] 1948 [illegible] *Cybernetics: Control and Communication in the Animal and the Machine* [illegible]

[illegible] “[illegible]”[35] [illegible]

[illegible]“[illegible]”[illegible]·[illegible]“[illegible]”[illegible]·[illegible]“[illegible]”[illegible]“[illegible]·[illegible]ENIAC[illegible]EDVAC[illegible]”[illegible][36]

[illegible]5[illegible]“[illegible]”[illegible]

> [illegible]“[illegible]”[illegible]“[illegible]”[illegible][37]

[illegible]“[illegible]”[38][illegible]*The Human Use of Human Beings: Cybernetics and Society*[illegible]1948[illegible]

[illegible]1951[illegible][39][illegible][40]

[illegible]“[illegible]”[41][illegible]1969[illegible]46[illegible]

[illegible]cyber[illegible]cyborg[illegible]cybernetic organism[illegible]cyberspace[illegible]cybercafe[illegible]cyberpunk[illegible]cybersex[illegible]cyber[illegible]e[illegible]

[illegible]·[illegible]·[illegible]EDVAC[illegible]Electronic Discrete Variable Automatic Computer[illegible]EDVAC[illegible]1945[illegible]

6[illegible]30[illegible]·[illegible]“[illegible]’[illegible]’[illegible]”[42][illegible]·[illegible]“[illegible]”[43]

[illegible]·[illegible]automaton[illegible][44][illegible]·[illegible]

> [illegible][45]

[illegible]·[illegible]1948[illegible]9[illegible]“[illegible]”[illegible]“[illegible]”[illegible]1948[illegible][46]

[illegible]·[illegible]“[illegible]……[illegible][47][illegible]Kinsey report[illegible]”[48] [illegible]·[illegible]DNA[illegible]

[illegible]Lisp[illegible]·[illegible]*Automata Studies*[illegible]1956[illegible]·[illegible]·[illegible]·[illegible]

[illegible]20[illegible]50[illegible]·[illegible]·[illegible]1955[illegible]·[illegible]1956[illegible]53[illegible]·[illegible][49]

[illegible]·[illegible]*The Computer and the Brain*[illegible]1958[illegible]·[illegible]·[illegible]·W. [illegible]*Theory of Self-Reproducing Automata*[illegible]1966[illegible]

[illegible]·[illegible]kinematic automata[illegible]

[illegible]·[illegible]·[illegible]——[illegible]cellular automata[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]·[illegible]29[illegible][50]

[illegible]20[illegible]70[illegible]·[illegible]·[illegible]1937—[illegible]“[illegible]”[illegible]Game of Life[illegible]8[illegible]2[illegible]3[illegible]0[illegible]1[illegible]4[illegible]4[illegible]3[illegible]“[illegible]”[illegible]

[illegible]·[illegible]“[illegible]”[illegible][51]
1974[illegible]“[illegible]”[illegible]
[52][illegible]1974[illegible]

[illegible][53]

[illegible]·[illegible]

1969[illegible]74[illegible]*Rechnender Raum*[illegible]“[illegible]”[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[54][illegible]“[illegible]”[illegible]

[illegible]

[illegible]137[illegible]45[illegible]37[illegible]1[illegible]2[illegible]

——[illegible]

[illegible][55][illegible]Maxwell’s Demon[illegible]·[illegible]·[illegible]1831—1879[illegible]

[illegible]·[illegible]·[illegible]1911—2008[illegible]“[illegible]”[illegible]it from bit[illegible]

> “[illegible]”[illegible][56]

[illegible]“[illegible]”[illegible][57]

[illegible]·[illegible]1953—[illegible]“[illegible]”[illegible]·[illegible]1930—1982[illegible]

1997[illegible]

〉[illegible]·[illegible]1902—1994[illegible]

〉[illegible]·[illegible]

〉[illegible]·[illegible]1909—1982[illegible]·[illegible]1941—[illegible]

〉[illegible]

[illegible]“[illegible]”[illegible]Turing principle[illegible]“[illegible]”[illegible]“[illegible]”[58][illegible]

MIT[illegible]·[illegible]1960—[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible][59][illegible]

> [illegible]……[illegible][60]

[illegible]

[illegible]Mathematica[illegible]·[illegible]1959—[illegible]2002[illegible]*A New Kind of Science*[illegible]“[illegible]……[illegible]”[illegible][61]

[illegible]

> [illegible]……[illegible][62]

[illegible]——[illegible]

[illegible]

[illegible]“[illegible]/[illegible]”[illegible]·[illegible]1596—1650[illegible]1641[illegible]

20[illegible]40[illegible]·[illegible]1943[illegible]

"[63]

·1900—1976 *The Concept of Mind* 1949

·1950 " "

·1942— "· "[64]

" " consciousness

·1966— "
"

" " " "

·1932— " "[65]

·1931—
1989 1994

1958
· ·
·1929— 1961

[illegible][66][illegible]

[illegible]·[illegible]

[illegible]1991[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]”[illegible]”[illegible]

[illegible]

> [illegible](1) [illegible](2) [illegible](3) [illegible][67]

[illegible]“[illegible]”[illegible]——[illegible]

[illegible]

[illegible]free will[illegible]

[illegible]·[illegible]1749—1827[illegible]*Essai Philosophique sur les Probabilites*[illegible]1814[illegible]

> [illegible][68]

[illegible]Laplace′s Demon[illegible]

[illegible]

[illegible]“[illegible]”[illegible]many-body problem[illegible]3[illegible]

[illegible]“[illegible]”[illegible]“[illegible]”[illegible]“[illegible]”[illegible]

[illegible]·[illegible]

> [illegible]……[illegible][69]

[illegible]·[illegible]“[illegible]”[illegible]

> [illegible]“[illegible]”[illegible][70]

[illegible]“[illegible]”[illegible]”[illegible]”[71]

[illegible]·[illegible]”[illegible]……[illegible]……”[illegible]”[illegible]”[illegible][72]

[illegible]

[illegible][73][illegible]“[illegible]”

□1□ □□□□□□□□□□□□□□□□□*Code: The Hidden Language of Computer Hardware and Software*□Microsoft Press□1999□□

□2□ □□□·□□·□□□□“Origins of Recursive Function Theory”□*Annals of the History of Computing*□ Vol. 3, NO.1 □Jan.1981□□61□

□3□ □□□·□□·□□□□*Introduction to Metamathematics*□D. Van Nostrand□1952□□358-360□

□4□ □□□□*Introduction to Metamathematics*□358□

□5□ □□□□*Introduction to Metamathematics*□300□376□

□6□ □□□□*Mathematical Logic*□John Wiley & Sons□1967□Dover□2002□□232□

□7□ □□B. Jack Copeland□*The Essential Turing*□Oxford University Press□2004□□40□□□61□

□8□ □□·□□□□*Computability and Unsolvability*□McGraw-Hill□1958□Dover□1982□□vii-viii□

□9□ □□·□□□□*Computability and Unsolvability*□vii□

□10□ □□·□□□□*Computability and Unsolvability*□70□

□11□ Raúl Rojas□“How to Make Zuse’s Z3 a Universal Computer”□*IEEE Annals of the History of Computing*□Vol. 20□No. 3□1998□□51-54□

□12□ □□·□□□“Systems of Logic Based on Ordinals”□*Proceedings of the London Mathematical Society*□*Series* 2□Vol. 45□1939□□172-173□

□13□ □□·□□□□“The Myth of Hypercomputation”□Christof Teuscher□ed., *Alan Turing*□*Life and Legacy of a Great Thinker*□Springer, 2004□□195–211□

□14□ □□C. Jack Copeland□“The Church- Tuning Thesis”□□□□□□□□□Stanford Encyclopedia of Philosophy□http://plato.stanford.edu/entries/church-turing□

□15□ □□□□□□Pamela McCorduc□□□*Machines Who Think A Personal Inquiry into the History and Prospects of Artificial Intelligence* [25th anniversersary edition□A. K. Peters□2004□89]□□□□□□□□□□□□□Manual Blum□□□□□□□□□□□□□□□□Rook McCulloch□□□*Collected Works of Warren S. McCulloch* [Intersystems Publications□1989□ Vol. I□ 31]□□Manual Blum□□□“Notes on McCorduc-Pitts’ *A Logical Calculus of the*

Ideas Immanent in Nervous Activity”□ □□□□□□□□□□□□James A. Anderson□Edward Rosenfeld□□*Talking Nets: An Oral History of Neural Networks* [MIT Press□1998□218]□□ Michael A. Arbib□□□□

[16] □□□Jack D. Cowan *Talking Nets*□104□

[17] □□□Jerome Y. Lettvin□*Talking Nets*□2□□□Jerome Y. Lettvin□“Warren and Walter”□*Collected Works of Warren S. McCulloch*□Vol. II□514-529□□□□□□□□□□□□□□□□□□□□□□□□□□

[18] Neil R. Smalheiser “Walter Pitts”□*Perspectives in Biology and Medicine*□Vol.43□No. 2 □Winter 2000□□218□

[19] Smalheiser□“Walter Pitts”□ 22□

[20] 4F□□□“□□□□□□□□□□□□□”□□□□□□——□□□

[21] □⑤□

[22] Tara H. Abraham□“Nicholas Rashevsky’s Mathematical Biophysics”□*Journal of the History of Biology*□Vol.37□No.2□Summer 2004□□333–385□

[23] □□·□□□□□“What is a Number□That a Man May Know it, and a Man, That he May Know a Number ?”□*Collected Words of Warren S. McCulloch*□Vol. IV□1226□

[24] □□□□□□□□□□□□□Michael A. Arbib□“Warren McCulloch’s Search for the Logic of the Nervous System”□*Perspectives in Biology and Medicine*□Vol.43□No.2□Winter 2000□□193–216□

[25] □□□Jack D. Cowan□*Talking Nets*□102□

[26] Arbib□“Warren McCulloch’ s Search for the Logic of the Nervous System”□202□

[27] Arbib□“Warren McCulloch’s Search for the Logic of the Nervous System□”199□

[28] Tara H. Abraham□“□Physio□logical Circuits□The Intellectual Origins of the McCulloch-Pitts Neural Networks□” *Journal of the History of the Behavioral Sciences*□Vol.38□No.1□Winter 2002□□19□

[29] □□□□□□□□□“A Logical Calculus in the Ideas Immanent in Nervous Activity”□*Bulletin of Mathematical Biophysics*□Vol. 5□1943□□129□□□□□ *Collected Works of Warren S. McCulloch*□Vol. I□357□

[30] □□□□Lloyd A. Jeffress□ed.□*Cerebral Mechanisms in Behavior: The Hixon Symposium*□John Wiley & Sons, 1951□□32–33□

[31] □□□□□“Mysterium Iniquitatis of Sinful Man Aspiring into the Place of God”□*The Scientific Monthly*□Vol. 80□No.1□Jan.1955□□36□□□*Collected Works of Warren S. McCulloch*□Vol. III□985□

[32] □□□38□

[33] Flo Conway and Jim Siegelman□*Dark Hero of the Information Age: In Search of Norbert Wiener, the Father of Cybernetics*□Basic Books□2005□□21□

[34] Conway and Siegelman□*Dark Hero*□155□

[35] December 27, 1948 issue, □□Conway and Siegelman□*Dark Hero*□182□

[36] □□□*Cybernetics: or Control and Communication in the Animal and the Machine* □John Wiley & Sons□1948□second edition□MIT Press□1961□□12□13□14□15□□□□□2□□

[37] □□□*Cybernetics*□120□

[38] □□□*Cybernetics*□144□

[39] Arbib□“Warren McCulloch’s Search for the Logic of the Nervous System□” 201-202□

[40] Conway and Siegelman□*Dark Hero*□222–229□

[41] Smalheiser□“Walter Pitts□”223□

[42] □·□□□□*First Draft of a Report on the EDVAC*□Moore School of Electrical Engineering□1945□□§4.1□

[43] □□□§4.2□

[44] William Aspray□“The Scientific Conceptualization of Information: A Survey”□*Annals of the History of Computing*□Vol. 7□No. 2□April, 1985□□133□

[45] Letter of November 29□1946□from Miklós Rédei□ed.□*John von Neumann: Selected Letters* □American Mathematical Society□2005□□278□

[46] □·□□□□“The General and Logical Theory of Automata”in Lloyd A. Jeffress□*Cerebral Mechanisms in Behavior: The Hixon Symposium*□John Wiley & Sons□1951□□1-41□

[47] [illegible]·[illegible]□□□□——□□□

[48] Letter of August 10□1949□quoted in Steve J. Heims□ *John von Neumann and Norbert Wiener: From Mathematics to the Technologies of Life and Death*□MIT Press□1980□□212□

[49] Heims□*John von Neumann and Norbert Wiener*□369-371□

[50] William Aspray□*John von Neumann and the Origins of Modem Computing*□MIT Press□1990□□203-204□

[51] □□□□□□□□□·□□□□□□*Wheels, Life, and Other Mathematical Amusements*□W. H. Freeman□1983□□

[52] 1974□1□21□□□□□□William Poundstone□*The Recursive Universe□Cosmic Complexity and the Limits of Scientific Knowledge*□William Morrow□1985□□24□

[53] Paul Rendell□“A Turing Machine in Conway’s Game of Life”□March 8□2001□ http://www.cs.ualberta.ca/~bulitko/F02/papers/tm_words.pdf □□□□http://rendell-attic.org /gol/tm.htm□

[54] □□□·□□□*Calculating Space*□□□*Rechnender Raum*□MIT Technical Translation□1970□□22□□□□□p16□□

[55] Tom Siegned□*The Bit and the Pendulum: From Quantum Computing to M Theory — The New Physics of Information*□John Wiley & Sons□2000□□Hans Christian von Baeyer□*Information: The New Language of Science*□Harvard University Press□2003□□Charles Seife□*Decoding the Universe: How the New Science of Information is Explaining Everything in the Cosmos from Our Brains to Black Holes* □Viking□2006□□

[56] □□·□□□□□·□□□“Information, Physics, Quantum: The Search for Links”□1989□ in Anthony J. G. Hey□ed.□*Feynman and Computation: Exploring the Limits of Computers* □Perseus Books□1999□□311□

[57] □□□314□□□□□□□□□□1988□□□□□□□

□58□ □□·□□□□The Fabric of Reality□Penguin Books□1997□□132-135

□59□ □□·□□□□*Programming the Universe: A Quantum Computer Scientist Takes on the Cosmos*□Alfred A. Knopf□2006□□46□53□

□60□ □□□54-55□

□61□ □□□·□□□□□□*A New Kind of Science*□Wolfram Media□2002□□538□

□62□ □□□720□721□

□63□ Arbib□"Warren McCulloch's Search for the Logic of the Nervous System"□213□

□64□ □□□·□□□□interview in Susan Blackmore□*Conversations on Consciousness: What the Best Minds Think About the Brain, Free Will, and What it Means to be Human*□Oxford University Press□2006□□81□

□65□ □□·□□□"Minds, Brains, and Programs□"from *The Behavioral and Brain Sciences*□Vol. 3□Cambridge University Press□1980□Republished in Douglas R. Hofstandter and Daniel Dennett□eds.□*The Mind's I: Fantasies and Reflections on Self and Soul*□Basic□Books□1981□□353-373□

□66□ □□·□□□□"Minds, Machines and Gödel"□*Philosophy*□Vol. 36□No.137□Apr.-Jul. 1961□□112-127□

□67□ □□□·□□□□*Consciousness Explained*□Back Bay Books□1991□□219□

□68□ □□□-□□·□□□□□*A Philosophical Essay on Probabilities*□□□Frederick Wilson Truscott □ Frederick Lincoln Emory□John Wiley & Sons□1902□Dover□1995□□

□69□ □□□□*Programming the Universe*□98□36□

□70□ □□□□*The Fabric of Reality*□238□

□71□ □□□336□339□

□72□ □□□□□□*A New Kind of Science*□739□741□750□

□73□ □□□□·□□□□□□□□□□·□□□□□□□□□□□□□□

[illegible]18[illegible]
[illegible]

[illegible]10[illegible]·[illegible]1900[illegible]20[illegible]

10[illegible]

[illegible][1]

[illegible]3[illegible]10[illegible]

[illegible]“[illegible]”[illegible]

[illegible]10[illegible]10[illegible]·[illegible]

[illegible]·[illegible]

[illegible][2]

[illegible]·[illegible]1919[illegible]·[illegible]2[illegible]·[illegible]2[illegible]

[illegible]9[illegible]2[illegible]5[illegible]8[illegible]

[illegible]2[illegible]

[illegible][3]

1933[illegible]

[illegible]1936[illegible]17[illegible]12[illegible]“[illegible]”[illegible][4]

1937[illegible]Simon & Schuster[illegible]*Men of Mathematics*[illegible]E. T. [illegible]1883—1960[illegible]

1939[illegible]·[illegible]8[illegible]“[illegible]4[illegible]……[illegible]”[5]

[illegible]1941[illegible]·[illegible]·[illegible]“[illegible]——[illegible]——[illegible]——[illegible]——[illegible]——[illegible]”[6]

[illegible][7]

[illegible]1946[illegible]1947[illegible]

[illegible]·[illegible]1902—1983[illegible]1948[illegible]“[illegible]”[illegible][8]

1948[illegible]10[illegible]

[illegible]10[illegible][9][illegible]

[illegible]

$$x^2 + ay^2 = s^2,\ x^2 - ay^2 = t^2,$$

[illegible]

$$x - 2y = 0$$

Diophantine relation

$$x - y + z + 1 = 0$$

[10]

$$x^n + y^n = z^n$$

1950

10

10 [11]

10

[12]

[13]

1950 · 0 *From Zero to Infinity* 1955 “ ” 1970 3 · 1976 1982 E. T. 1993 *Julia: A Life in Mathematics* 1996

1958 1959 1960 · · 1926— 1959 “ ” The Decision Problem for Exponential Diophantine Equations 1961 [14]

“ ”

10 “ · ” [15]

20 60 10 1969 *Studies in Number Theory* 40

> $r = S^t$ 10 [16]

20 60 10 “ · ” [17]

· 1947 17 1966 1970 LOMI

10 1965 1939—1981 “ 10 ” “

[illegible]”[illegible]10[illegible]“[illegible]……[illegible]”[18]

[illegible]10[illegible]“[illegible]10[illegible]”[19][illegible]1961[illegible]

1970[illegible]·[illegible]22[illegible]1[illegible]10[illegible]·[illegible]“[illegible]22[illegible]! “[20]

[illegible]·[illegible]1971[illegible]

[illegible]·[illegible][21][illegible]10[illegible]1982[illegible]1974[illegible]5[illegible]”[illegible]“[illegible]Diophantine Equations: Positive Aspects of a Negative Solution[illegible][22][illegible]

[illegible]10[illegible]·[illegible]1976[illegible]1982[illegible]*Ladies Home Journal*[illegible][23]

1983[illegible]·[illegible]“[illegible]”[illegible]

1984[illegible]·[illegible]65[illegible]·[illegible]1995[illegible]83[illegible]

[illegible]·[illegible]80[illegible]

1993[illegible]10[illegible]*Hilbert’s Tenth Problem*[illegible]10[illegible]

[illegible]10[illegible]5[illegible]“[illegible]”[illegible][24]

[illegible]

[illegible]

17[illegible]·[illegible]

[illegible]

---

[1]Ben H. Yandell□*The Honors Class: Hilbert’s Problems and Their Solvers*□A. K. Peters□2002□□406□

[2]□□□□· □□□*Julia*□*A Life in Mathematics*□Mathematical Association of America□1996□□ 3□□□□□□□“The Autobiography of Julia Robinson□”□□*More Mathematical People* □Academic Press□1990□□262–280□□□□

[3]□□□□□*Julia: A Life in Mathematics*□9□

[4]□□□21□

[5]□□□□□*Julia: A Life in Mathematics*□31□35□

[6]□□□33□

[7]□□□43□

[8]□□□□· □□□□□□· □□□□“Julia Bowman Robinson□1919—1985□”□in *A Century of Mathematics in America*□Part III □Amencan Mathematical Society□1989□□410□

[9]□□□·□□□□“Existential Definability in Arithmetic”□*Transactions of the American Mathematical Society*□Vol. 72 □1952□□437-449□Also published in Julia Robinson□*The Collected Works of Julia Robinson*□ed Solomon Feferman □American Mathematical Society□1996□□47-59□

[10] □□□□“Existential Definability in Arithmetic”□438□

[11] □□·□□□□Foreword to Yuri V. Matiyasevich□*Hilbert’s Tenth Problem*□MIT Press□1993□□xiii□

[12] □□□□·□□□*Julia: A Life in Mathematics*□61□

[13]□□·□□□□*Hilbert’ s Tenth Problem* □□□xiv□

[14]□□□—□□□—□□□□“The Decision Problem for Exponential Diophantine Equations”□*Annals of Mathematics*□Vol. 74□No.3□Nov. 1961□□425-436□□□□□ *The Collected Works of Julia Robinson*□77-88□

[15]□□□□*Hilbert’s Tenth Problem* □□□xiii□

[16]□□□·□□□□“Diophantine Decision Problems”□W. J. LeVeque□ed.□*Studies in Number Theory*□Mathematical Association of America□1969□□107□□□□ *Collected Works of Julia Robinson*□176□

[17]□□□□ *Hilbert’s Tenth Problem* □□□xiii□

[18]□□·□□□□□□□“My Collaboration with Julia Robinson□” *The Mathematical Intelligencer*□Vol. 14 □No. 4□1992□□38-45□□□□□□□□·□□□*Julia: A Life in Mathematics*□99-116□

[19]□□·□□□□□□□“My Collaboration with Julia Robinson□” *The Mathematical Intelligencer*□Vol. 14 □No. 4□1992□□38-45□□□□□□□□·□□□*Julia: A Life in Mathematics*□99-116□

[20]□□□□·□□□*Julia: A Life in Mathematics*□73□

[21]Martin Davis and Reuben Hersh□“Hilbert’ s 10th Problem”□*Scientific American*□Vol. 229□No.5□Nov. 1973□□84-91□

[22]□□□□□□□□□□□□□□□“Hilbert’s Tenth Problem. Diophantine Equations: Positive Aspects of a Negative Solution”□in Felix E. Browder□ed.□*Mathematical Developments Arising from Hilbert Problems*□American Mathematical Society□1976□□Vol. 2□323-378□□□□□*Collected Works of Julia Robinson*□269-324□

[23]□□□□□*Julia: A Life in Mathematics*□81□

[24]□□□□□□□*Hilbert’s Tenth Problem*□93□

□□□□

□□□□

□□·□□□□□□□□·□□□□□□□□www.turing.org.uk□
□□□□□□□□□□□□□□www.AlanTuring.net□
□□□□□□□□□□□www.turingarchive.org□

□□

Ackermann, Wilhelm. *Solvable Cases of the Decision Problem.* North-Holland Publishing Company, 1962.

Agar, John. *Turing and the Universal Machine: The Making of the Modern Computer.* Icon Books, 2001.

Arbib, Michael A. *Brains, Machines, and Mathematics.* McGraw-Hill, 1964.

Aspray, William. *John von Neumann and the Origins of Modern Computing.* MIT Press, 1990.

Beth, Evert W. *The Foundations of Mathematics: A Study in the Philosophy of Science,* second edition. North-Holland, 1964; Harper & Row, 1966.

Boolos, George S., John P. Burgess, and Richard C. Jeffrey. *Computability and Logic,* fourth edition. Cambridge University Press, 2002.

Börger, Egon, Erich Grädel, and Yuri Gurevich. *The Classical Decision Problem.* Springer, 1997, 2001.

Boyer, Carl B. *A History of Mathematics,* second edition revised by Uta C. Merzbach. John Wiley & Sons, 1989.

Cantor, Georg. *Contributions to the Founding of the Theory of Transfmite Numbers,* translated and with an

introduction by Philip E. B. Jourdain. Open Court, 1915; Dover, 1955.

Carpenter, B. E. and R. W. Doran. *A. M. Turing's ACE Report of 1946 and Other Papers.* MIT Press, 1986.

Ceruzzi, Paul E. *A History of Modern Computing,* second edition. MIT Press, 2003.

Ceruzzi, Paul E. *Reckoners: The Prehistory of the Digital Computer, from Relays to the Stored Program Concept, 1935-1945.* Greenwood Press, 1983.

Chaitin, Gregory. *Meta Math! The Quest for Omega.* Vintage Books, 2005.

Chaitin, Gregory. *Thinking about Gödel and Turing: Essays on Complexity, 1970-2007.* World Scientific, 2007.

Church, Alonzo. *The Calculi of Lambda-Conversion.* Princeton University Press, 1941.

Church, Alonzo. *Introduction to Mathematical Logic.* Princeton University Press, 1956.

Clark, Andy and Peter Millican, eds. *Connectionism, Concepts, and Folk Psychology: The Legacy of Alan Turing,* Vol. 2. Clarendon Press, 1996.

Conway, Flo and Jim Siegelman. *Dark Hero of the Information Age: In Search of Norbert Wiener, the Father of Cybernetics.* Basic Books, 2005.

Copeland, B. Jack. *Alan Turing's Automatic Computing Engine.* Oxford University Press, 2005.

Copeland, B. Jack. *The Essential Turing.* Oxford University Press, 2004.

Dauben, Joseph Warren. *Georg Cantor: His Mathematics and Philosophy of the Infinite.* Princeton University Press, 1979.

Davis, Martin. *Computability and Unsolvability.* McGraw-Hill, 1958. Enlarged edition, Dover, 1982.

Davis, Martin. *The Universal Computer: The Road from Leibniz to Turing.* W. W. Norton, 2000. Published in

paperback under the title *Engines of Logic: Mathematicians and the Origin of the Computer.*

Davis, Martin, ed. *The Undecidable: Basic Paper on Undecidable Propositions, Unsolvable Problems and Computable Functions.* Raven Press, 1965.

Dawson, John W. Jr. *Logical Dilemmas: The Life and Work of Kurt Gödel.* A. K. Peters, 1997.

DeLong, Howard. *A Profile of Mathematical Logic.* Addison-Wesley, 1970; Dover, 2004.

Dennett, Daniel. *Consciousness Explained.* Back Bay Books, 1991.

Devlin, Keith. *Mathematics: The New Golden Age.* Penguin, 1988.

Enderton, Herbert B. *A Mathematical Introduction to Logic,* second edition. Harcourt, 2001.

Epstein, Richard L. and Walter A. Carnielli. *Computability: Computable Functions, Logic, and the Foundations of Mathematics,* second edition. Wadsworth, 2000.

Eves, Howard. *Foundations and Fundamental Concepts of Mathematics,* third edition. PWS-Kent, 1990; Dover, 1997.

Ewald, William. *From Kant to Hilbert: A Source Book in the Foundations of Mathematics,* in two volumes. Clarendon Press, 1996.

Franzén, Torkel. *Gödel's Theorem: An Incomplete Guide to Its Use and Abuse.* A. K. Peters, 2005.

Gödel, Kurt. *Collected Works, Volume I: Publications 1929-1936,* edited by Solomon Feferman et. al. Oxford University Press, 1986.

Gödel, Kurt. *Collected Works, Volume II: Publications 1938-1974,* edited by Solomon Feferman et. al. Oxford University Press, 1990.

Gödel, Kurt. *Collected Works, Volume III: Unpublished Essays and Lectures,* edited by Solomon Feferman et. al. Oxford University Press, 1995.

Goldstein, Rebecca. *Incompleteness: The Proof and Paradox of Kurt Gödel.* W. W. Norton, 2005.

Goldstern, Martin and Haim Judah. *The Incompleteness Phenomenon: A New Course in Mathematical Logic.* A. K. Peters, 1995.

Grattan-Guiness, I. *The Search for Mathematical Roots, 1870-1940: Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell to Gödel.* Princeton University Press, 2000.

Gray, Jack and Keith Thrower. *How the Turing Bombe Smashed the Enigma Code.* Speedwell, 2001.

Heath, Sir Thomas L. *Diophantus of Alexandria: A Study in the History of Greek Algebra,* second edition. Cambridge University Press, 1910; Dover, 1964.

Heims, Steve J. *John von Neumann and Norbert Wiener: From Mathematics to the Technologies of Life and Death.* MIT Press, 1980.

Herkin, Rolf, ed. *The Universal Turing Machine: A Half-Century Introduction.* Oxford University Press, 1988. Second edition, Springer-Verlag, 1995.

Hilbert, David and Wilhelm Ackermann. *Grundzüge der Theoretischen Logik.* Springer, 1928. Second edition, Springer, 1938; Dover, 1946.

Hilbert, David and Wilhelm Ackermann. *Principles of Mathematical Logic* (translation of the second German edition of *Grundzüge der Theoretischen Logik*). Chelsea, 1950.

Hodges, Andrew. *Alan Turing: The Enigma.* Simon & Schuster, 1983.

Hodges, Andrew. *Turing: A Natural Philosopher.* Phoenix, 1997.

Hofstadter, Douglas R. *Gödel, Escher, Bach: an Eternal Golden Braid.* Basic Books, 1979.

Hunter, Geoffrey. *Metalogic: An Introduction to the Metatheory of Standard Eirst Order Logic.* University of California Press, 1971.

Kleene, Stephen Cole. *Introduction to Metamathematics.* Van Nostrand, 1952.

Kleene, Stephen Cole. *Mathematical Logic.* John Wiley & Sons, 1967; Dover, 2002.

Kline, Morris. *Mathematics: The Loss of Certainty.* Oxford University Press, 1980.

Kneale, William and Martha Kneale. *The Development of Logic.* Clarendon Press, 1962.

Leavitt, David. *The Man Who Knew Too Much: Alan Turing and the Invention of the Computer.* W. W. Norton, 2006.

Levin, Janna. *A Madman Dreams of Turing Machines.* Alfred A. Knopf, 2006.

Lewis, C. I. *A Survey of Symbolic Logic.* Univesity of California Press, 1918; Dover, 1960.

Mancosu, Paolo. *Erom Brouwer to Hilbert: The Debate on the Foundations of Mathematics in the 1920s.* Oxford University Press, 1998.

Matiyasevich, Yuri M. *Hilbert's Tenth Problem.* MIT Press, 1993.

Mendelson, Elliott. *Introduction to Mathematical Logic,* fourth edition. Chapman & Hall, 1997.

Millican, Peter and Andy Clark, eds. *Machines and Thought: The Legacy of Alan Turing,* Vol. 1. Clarendon Press, 1996.

Niven, Ivan. *Numbers: Rational and Irrational.* Mathematical Association of America, 1961.

Ore, Oystein. *Number Theory and Its History.* McGraw-Hill, 1948; Dover, 1988.

Peano, Giuseppe. *Selected Works of Giuseppe Peano,* translated and edited by Hubert C. Kennedy. George Allen & Unwin, 1973.

Penrose, Roger. *The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics.* Oxford University Press, 1989.

Petzold, Charles. *Code: The Hidden Language of Computer Hardware and Software.* Microsoft Press, 1999.

Phillips, Esther R., ed. *Studies in the History of Mathematics (MAA Studies in Mathematics, Volume 26).* Mathematical Association of America, 1987.

Prager, John. *On Turing.* Wadsworth, 2001.

Quine, Willard Van Orman. *Mathematical Logic,* revised edition. Harvard University Press, 1951, 1981.

Reid, Constance. *Hilbert.* Springer-Verlag, 1970, 1996.

Reid, Constance. *Julia: A Life in Mathematics.* Mathematical Association of America, 1996.

Robinson, Julia. *The Collected Works of Julia Robinson,* edited by Solomon Feferman. American Mathematical Society, 1996.

Russell, Bertrand. *Introduction to Mathematical Philosophy,* second edition. George Allen & Unwin, 1920; Dover, 1993.

Russell, Bertrand. *The Principles of Mathematics.* Cambridge University Press, 1903. W. W. Norton, 1938.

Shanker, S. G., ed. *Gödel's Theorem in Focus.* Routledge, 1988.

Sipser, Michael. *Introduction to the Theory of Computation.* PWS, 1997.

Teuscher, Christof, ed. *Alan Turing: Life and Legacy of a Great Thinker.* Springer, 2004.

Tiles, Mary. *The Philosophy of Set Theory: An Historical Introduction to Cantor's Paradise.* Basil Blackwell, 1989; Dover, 2004.

Turing, A. M. *Collected Works of A.* M. *Turing: Mathematical Logic,* edited by R. O. Gandy and C. E. M. Yates. Elsevier, 2001.

Turing, A. M. *Collected Works of A. M. Turing: Mechanical Intelligence,* edited by D. C. Ince. North-Holland, 1992.

Turing, A. M. *Collected Works of A. M. Turing: Morphogenesis,* edited by P. T. Saunders. North-Holland, 1992.

Turing, A. M. *Collected Works of A. M. Turing: Pure Mathematics,* edited by J. L. Britton. North-Holland, 1992.

van Atten, Mark. *On Brouwer.* Wadsworth, 2004.

van Heijenoort, Jean. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931.* Harvard University Press, 1967.

von Neumann, John. *The Computer and the Brain.* Yale University Press, 1958.

Wang, Hao. *Popular Lectures on Mathematical Logic.* Van Nostrand Reinhold, 1981; Dover, 1993.

Wang, Hao. *Reflections on Kurt Gödel.* MIT Press, 1987.

Whitehead, Alfred North and Bertrand Russell. *Principia Mathematics to *56.* Cambridge University Press, 1962.

Whitemore, Hugh. *Breaking the Code.* Fireside Theatre, 1987.

Wiener, Norbert. *Cybernetics, or Control and Communication in the Animal and the Machine.* John Wiley & Sons, 1948. Second edition, MIT Press, 1961.

Wilder, Raymond L. *Introduction to the Foundations of Mathematics.* John Wiley & Sons, 1952.

Wolfram, Stephen. *A New Kind of Science.* Wolfram Media, 2002.

Yates, David M. *Turing's Legacy: A History of Computing at the National Physical Laboratory 1945-1995.* Science Museum (London), 1997.